

Signals and Systems

Introduction

This course is about signals and their processing by systems. It involve

- Modeling of signals by mathematical functions
- Modeling of systems by mathematical equations
- Solution of the equations when excited by the functions

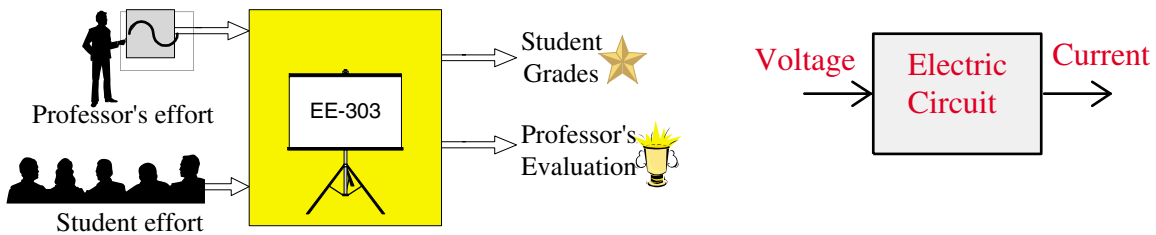
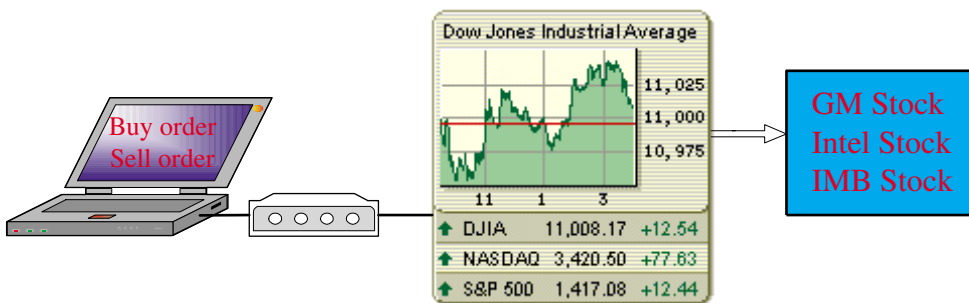
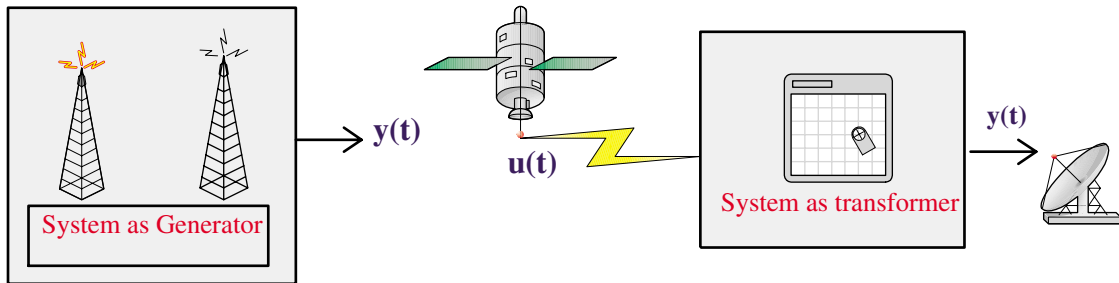
What are Signals?

A Signal is something that represent information. Our world is full of signals, both natural and man-made. Examples are:

- Variation in air pressure when we speak
- Voltage waveform in a circuit
- The periodic electrical signals (EKG) generated by the heart
- Stock prices of Intel
- No. of yawns in EE-303 through the quarter

What are systems?

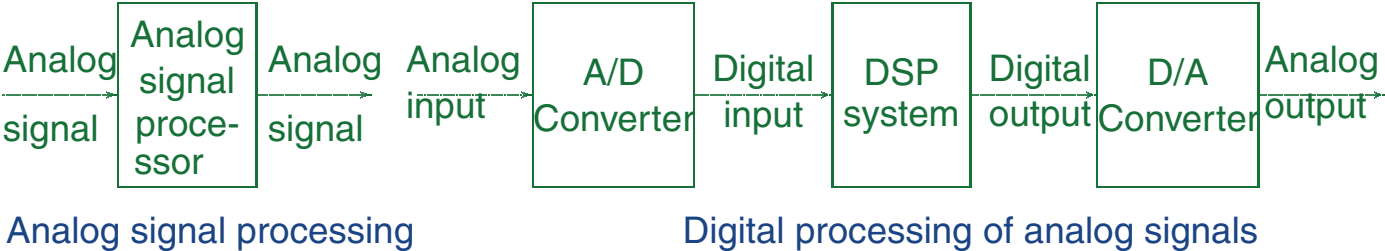
A system is a generator of signals or it is a transformer of signals



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What is Signal Processing?

Signal processing involves enhancing, extracting, storing and transmitting useful information. Electrical signals are best suited for such manipulations. It is common to convert signals to electrical form for processing. Two conceptual schemes for signal processing are shown below.



Why study Signals and Systems?

(What is the point of EE-303?)

Signals and systems are fundamental to all of engineering! Steps involved in engineering are:

- Model system: Involves writing a mathematical description of input and output signals
- Analyze system: Study of the various signals associated with the system
- Design system: Requires deciding on a suitable system architecture, as well as finding suitable system parameters
- Implement system/test system: Check system, and the input and output signals, to see that the performance is satisfactory.

Signal Types

The value of a signal, at any instant, corresponds to its (instantaneous) amplitude. Time may assume a continuum of value t , or discrete values, nt_s , where t_s is a sampling interval, and n is an integer. The amplitude may also assume a continuum of values or be quantized to a finite number or discrete levels between extremes. This results in four kinds of signals:

Continuous-time Signal (CT) Analog

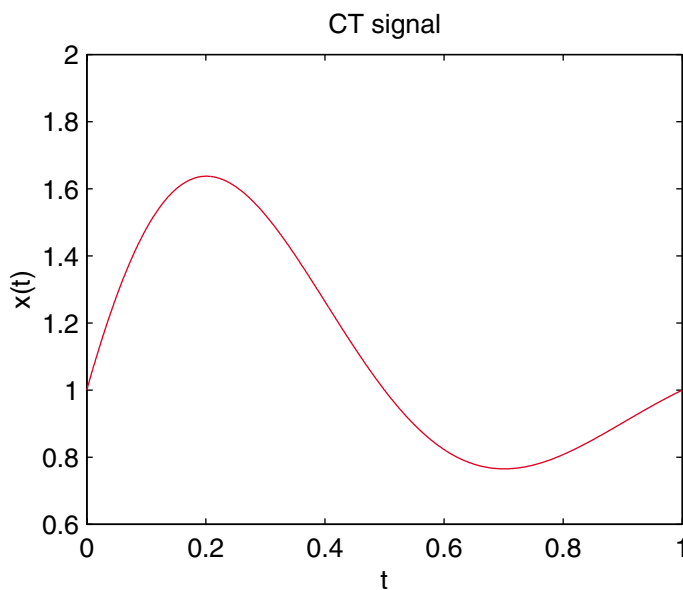
It is a function of a continuous-time variable. The air pressure variation caused by the vibration of the speaker diaphragm is an analog signal because the pressure variation is a continuous function of time.

Example 1.1

let $x(t) = 1 + e^{-2t} \sin(2\pi t)$. In MATLAB we use the following commands to plot this signal.

```
%chs1_ex1.m
t=0:0.001:1; % CT index(0, 1)
x = 1+ exp(-2*t).*sin(2*pi*t); %function x(t)
plot(t,x, 'r') % CT plot
xlabel('t'),ylabel('x(t)'),title('CT signal')
```

The graph of this CT signal is



Note that the signal may not be continuous in amplitude.

Discrete-time Signal (DT) Sampled

A discrete signal is defined at only certain instants of time. Between these discrete-time instants the value of the signal may be zero, or undefined. **A discrete-time signal is an ordered sequence of numbers.** The sampling intervals are usually the same.

Example 1.2

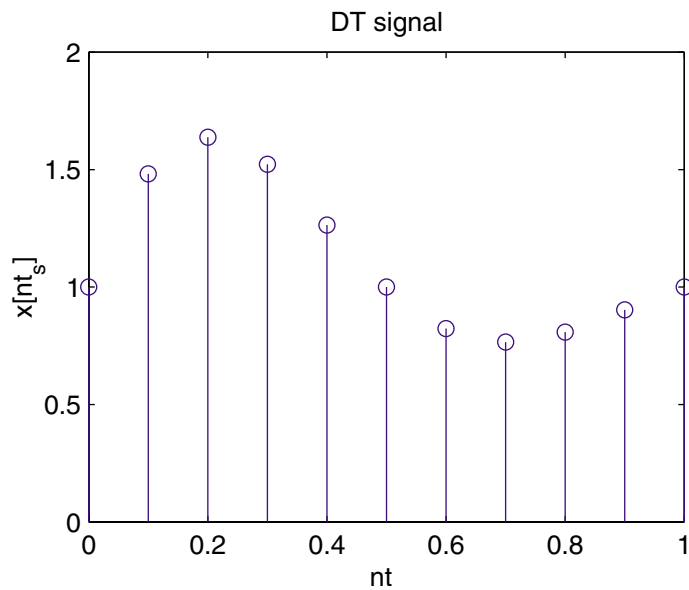
For $x[nt_s] = 1 + e^{-2nt_s} \sin(2\pi nt_s)$, we use the **stem** command to plot the DT signal.

```

%chs1_ex2.m
t_s = 0.1; % Sampling interval
nt_s=(0:1:10)*t_s;%DT (0.1 sampling interval)
x = 1+ exp(-2*nt_s).*sin(2*pi*nt_s);
stem(nt_s, x), % Sampled signal
xlabel('nt_s'),ylabel('x[nt_s]'),
title('DT signal')

```

The graph of this DT signal is



Quantized Signal

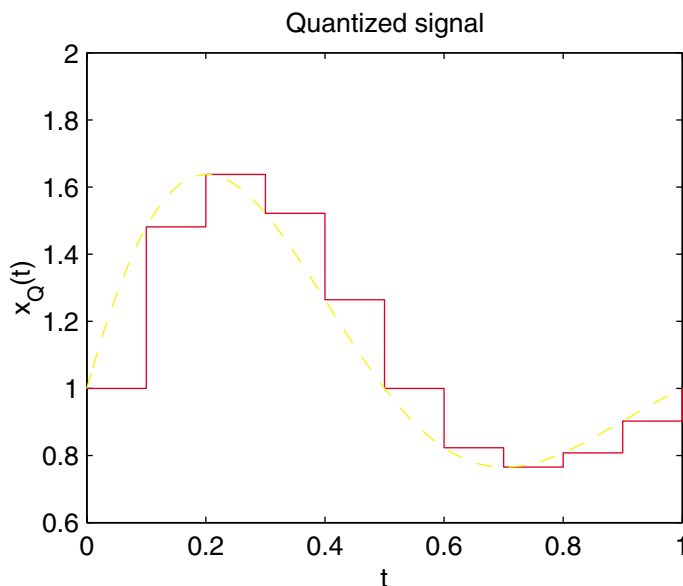
In quantize signal the amplitude assumes a continuum of values at continuous time intervals.

Example 1.3

For $x(t) = 1 + e^{-2t} \sin(2\pi t)$, we use the **stairs** command to plot the quantize signal.

```
nt_s=0:0.1:1;    %(chs1_ex3.m)    % QT
x = 1+ exp(-2*nt_s).*sin(2*pi*nt_s);
stairs(t, x, 'r') % Quantized signal
xlabel('t'),ylabel('x_Q(t)')
title('Quantized signal')
```

The graph of this quantize signal is



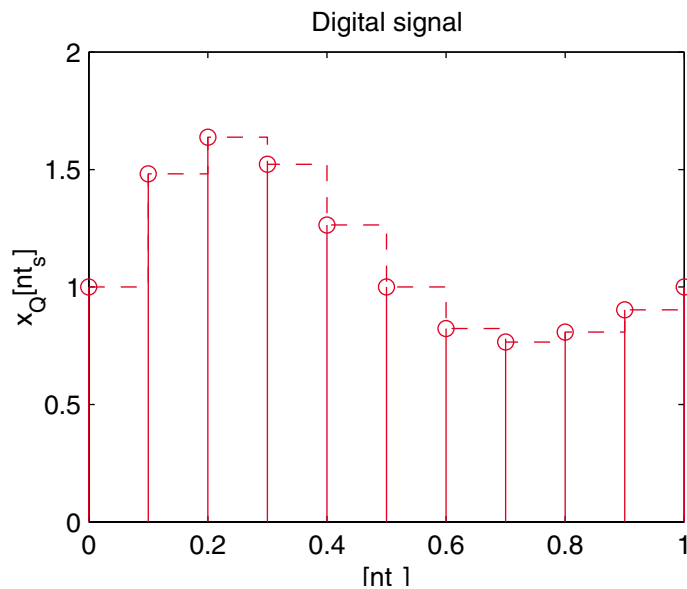
Digital signal

In a digital signal the amplitude is quantized to a finite number of discrete levels between its extremes.

Example 1.4

For $x[nt_s] = 1 + e^{-2nt_s} \sin(2\pi nt_s)$, we can use **stem** and **stairs** command.

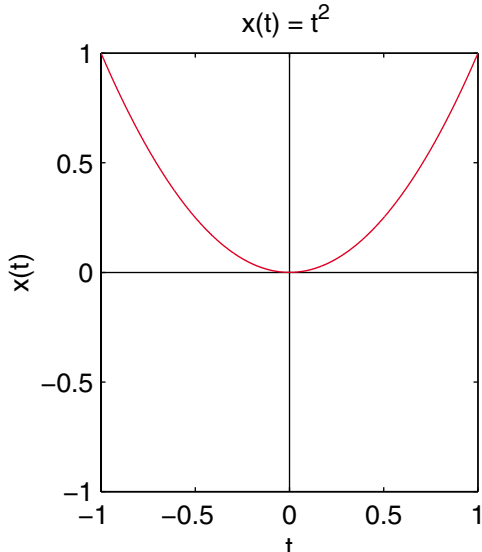
```
%ch1_ex4.m
t_s = 0.1; nt_s=(0:1:10)*t_s; % Sampling time
x = 1+ exp(-2*nt_s).*sin(2*pi*nt_s);
stem(t, x, 'r'), hold
stairs(t, x,'--r'), hold off
xlabel('[nt_s]'), ylabel('x_Q[nt_s]')
title('Digital signal')
```



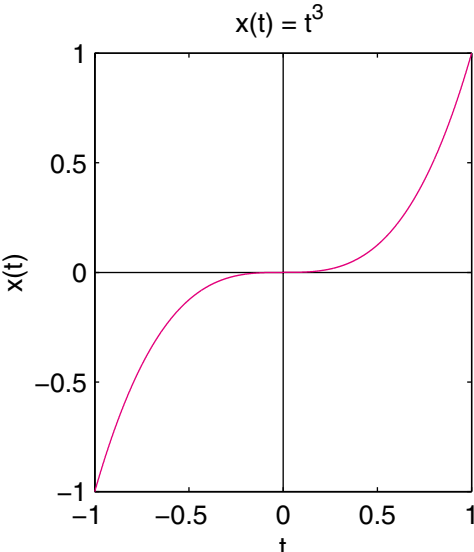
Even and Odd Signals

A signal $x(t)$ is even if $x(-t) = x(t)$ for all t . signal for $t < 0$ and signal for $t > 0$ are symmetric about the vertical axis passing through the origin at $t = 0$. i.e., they are mirror image of each other.

A signal $x(t)$ is odd if $x(-t) = -x(t)$ for all t . signal for $t < 0$ and signal for $t > 0$ are asymmetric about the origin.



Example of an even signal



Example of an odd signal

Any signal can be written as a sum of its even and odd parts, i.e.

$$x(t) = x_e(t) + x_o(t)$$

where

$$x_e(t) = x_e(-t)$$

and

$$x_o(t) = -x_o(-t)$$

Replacing t for $-t$ in $x(t)$, we have

$$x(-t) = x_e(-t) + x_o(-t)$$

or

$$x(-t) = x_e(t) - x_o(t)$$

Solving for $x_e(t)$ and $x_o(t)$, we obtain

$$x_e(t) = \frac{1}{2} [x(t) + x(-t)]$$

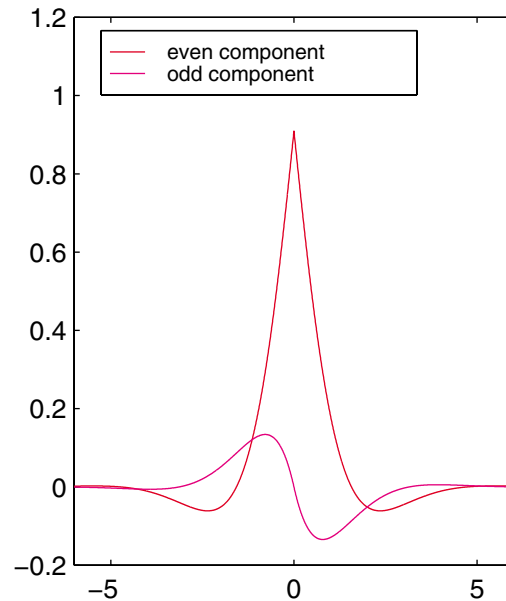
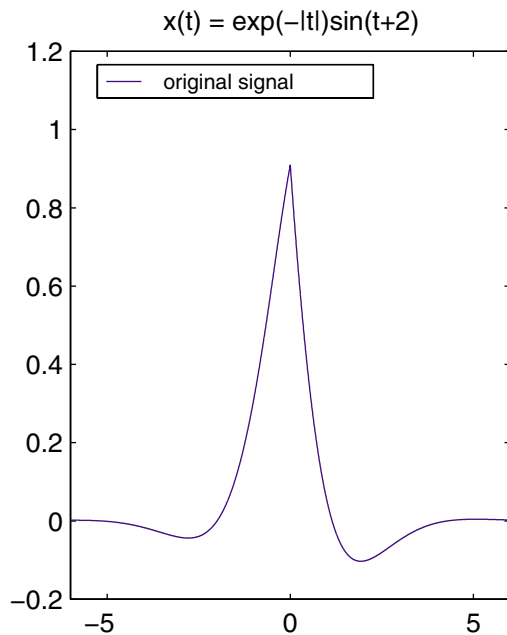
and

$$x_o(t) = \frac{1}{2} [x(t) - x(-t)]$$

Example 1.5

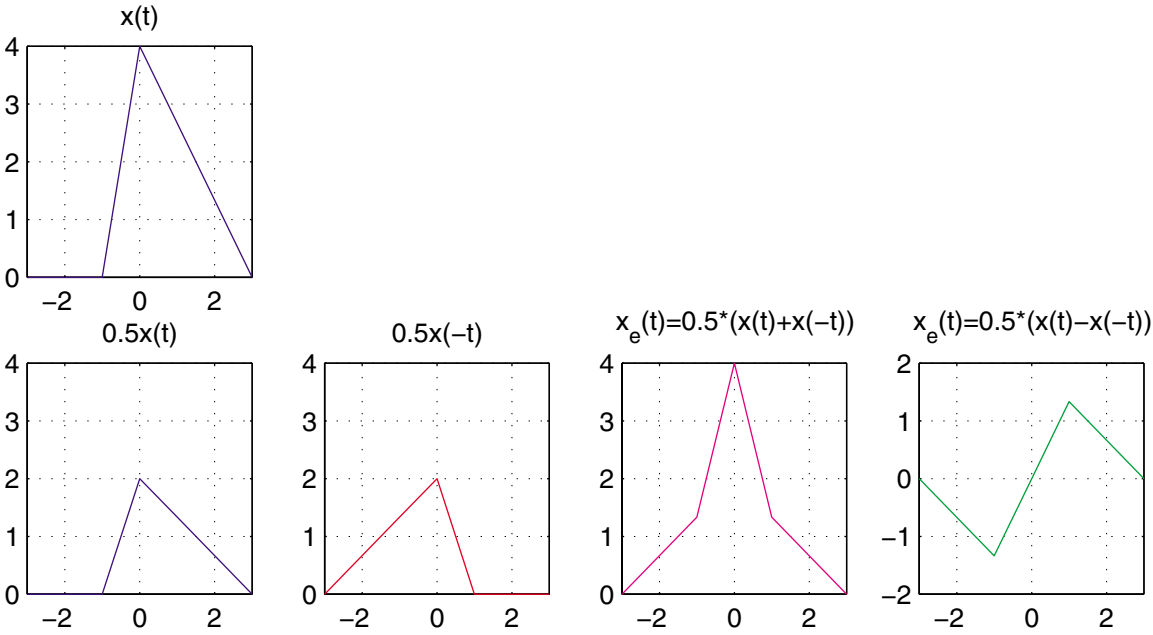
Given $x(t) = e^{-|t|} \sin(t+2)$, Use MATLAB to obtain the even and odd parts of $x(t)$.

```
%chs1_ex5.m
t=-6:.005:6;
x1 = exp(-abs(t)).*sin(t+2);
subplot(1,2,1), plot( t, x1)
legend('original signal',2)
title('x(t) = exp(-|t|)sin(t+2)')
x2 = exp(-abs(-t)).*sin(-t+2);
xe = 0.5*(x1+x2); xo = 0.5*(x1-x2);
subplot(1,2,2), plot(t, xe, 'r', t, xo, 'm')
legend('even component', 'odd component',2)
```



Example 1.6 (chs1_ex6.m)

Find the even and the odd parts of the signal $x(t)$ shown below. The signal $0.5x(t)$, and $0.5x(-t)$ are drawn as shown. The even part given by $x_e(t) = 0.5[x(t) + x(-t)]$, and the odd part given by $x_o(t) = 0.5[x(t) - x(-t)]$ are as shown.



Even and odd functions have the following properties:

- The sum of two even functions is even.
- The sum of two odd functions is odd.
- The sum of an even function and an odd function is neither even or odd
- The product of two even functions is even.
- The product of two odd functions is even. The product of an even function and an odd function is odd.

Periodic and nonperiodic signals

A continuous-time signal is periodic if and only if

$$x(t + T) = x(t) \quad \text{for all } t$$

where $T = T_0, 2T_0, 3T_0, \dots$ T_0 which is the duration of one complete cycle is called the fundamental period. The number of cycle per second or frequency is then

$$f = \frac{1}{T} \quad \text{Hz}$$

or the angular frequency is

$$\omega = 2\pi f \quad \text{rad/s}$$

Any other signal not satisfying the above condition is called *aperiodic* signal. A sine wave is an example of a periodic function. Consider the sine wave

$$x(t) = \sin(\omega t + \theta)$$

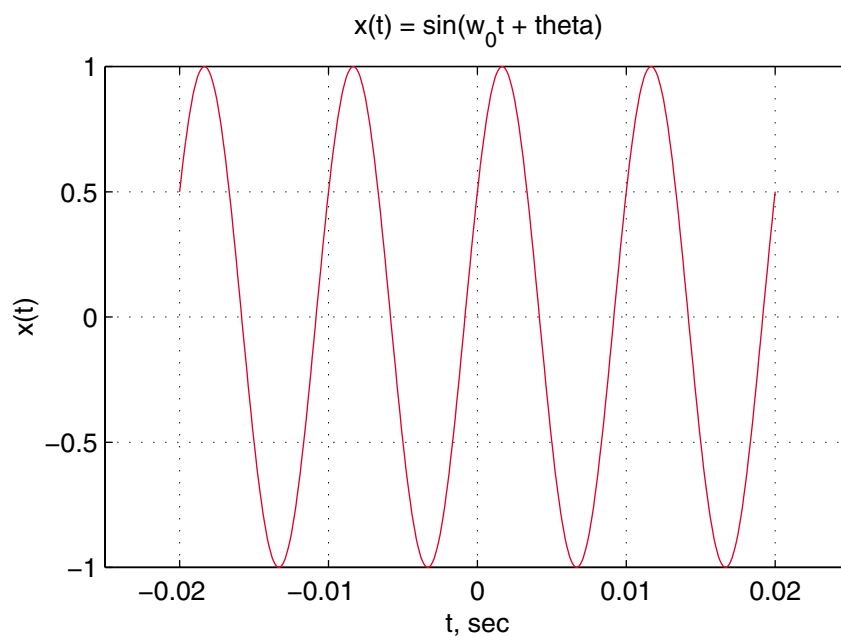
Example 1.7

Use MATLAB to plot $x(t)$, if $\omega = 200\pi$ rad/s, and $\theta = \frac{\pi}{6}$ rad. The period is $T = \frac{2\pi}{\omega} = \frac{2\pi}{200\pi} = 0.01$ s.

We use the following command

```
%chs1_ex7.m
w_0 = 200*pi;  theta = pi/6;
T = 2*pi/w_0;  % period
t = -2*T:T/100:2*T; % time range -2T to 2T
x = sin(w_0*t + theta);
plot(t, x), grid
xlabel('t'), ylabel('x(t)')
title('x(t) = sin(w_0t + theta)')
axis([-0.025 0.025 -1 1])
```

The result is



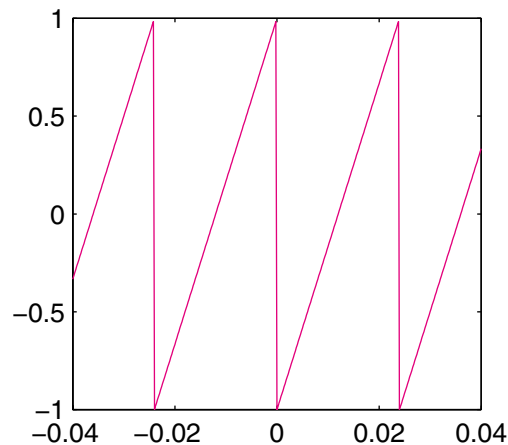
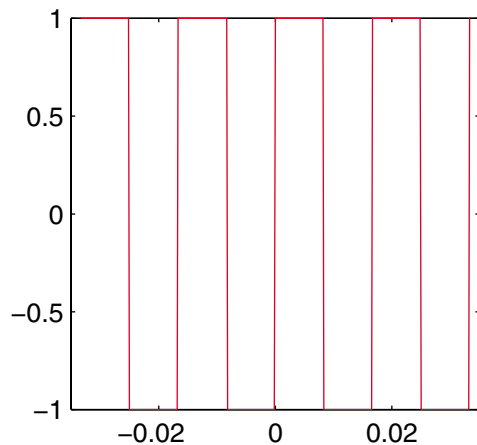
The signal processing toolbox has many function for plotting other periodic signals such as square and sawtooth. **square**(ωt) generates a square wave with peaks of $+1$ to -1 , and period $\frac{2\pi}{\omega}$. **square**(ωt , **Duty**) generates a square wave with the duty cycle. Duty is a constant between 0 and 1, specifying the fraction of each period for which the signal is positive. The command **sawtooth**(ωt) and **sawtooth**(ωt , **WIDTH**) generates a sawtooth wave. WIDTH is a scaler parameter between 0 and 1 which specifies the fraction between 0 and 2π at which the maximum occurs.

Example 1.8

Use MATLAB to generate a square wave of frequency $f = 60$ Hz, and a sawtooth wave of frequency $\omega_0 = 100\pi$ rad/s. We use the following commands

```
%chs1_ex8.m
T = 1/60;           % period sec.
t = -2*T:T/100:2*T; % time range
x1 = square(2*pi*60*t);
subplot(1,2,1), plot(t, x1, 'r')
w_0 = 100*pi; T = 2*pi/w_0;
x2 = sawtooth(w_0*t);
t = -2*T:T/100:2*T;
subplot(1,2, 2), plot(t, x2, 'm')
subplot(111)
```

The result is as shown.



The sum of two or more continuous-time periodic waves is periodic if the ratio of each pair of individual frequencies is a rational fraction, i.e., their frequencies are commensurable.

A discrete-time signal is periodic if and only if

$$x[n + N] = x[n] \quad \text{for all integers } n$$

where $N = N_0, 2N_0, 3N_0, \dots$. The integer N_0 is the fundamental period, with the angular frequency

$$\Omega = \frac{2\pi}{N_0} \quad \text{rad/s}$$

Operation on Signals

Typical operations on signals are three transformations in time domain and three transformation in amplitude. These transformations are especially useful in Fourier series applications.

Time Shift

If $x(t)$ is a continuous function, the time-shifted signal is defined as $y(t) = x(t - t_0)$. If $t_0 > 0$, the signal is shifted to the right, and if $t_0 < 0$, the signal is shifted to the left.

Example 1.9 (chs1_ex9.m)

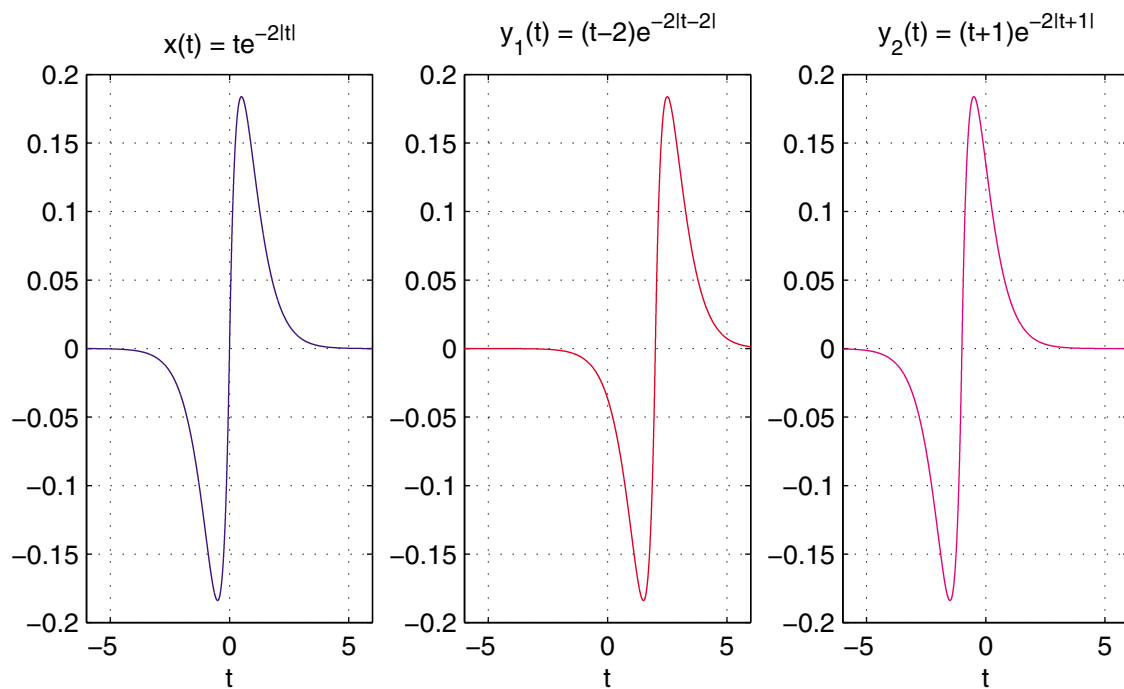
Consider $x(t) = te^{2|t|}$, we wish to plot the time-shifted signal $y_1(t) = (t - 2)e^{2|t-2|}$, and $y_2(t) = (t + 1)e^{2|t+1|}$.

We use the following commands

```

%chs1_ex9.m
t=-6:.001:6;
x =t.*exp(-2*abs(t));
y1 = (t-2).*exp(-2*abs(t-2));
y2 = (t+1).*exp(-2*abs(t+1));
subplot(1,3,1), plot(t, x, 'b'), grid
title('x(t) = te^{-2|t|}'),xlabel('t')
subplot(1,3,2), plot(t, y1,'r'), grid
title('y_1(t) = (t-2)e^{-2|t-2|}'),xlabel('t')
subplot(1,3,3), plot(t, y2, 'm'), grid
title('y_2(t) = (t+1)e^{-2|t+1|}'),xlabel('t')

```



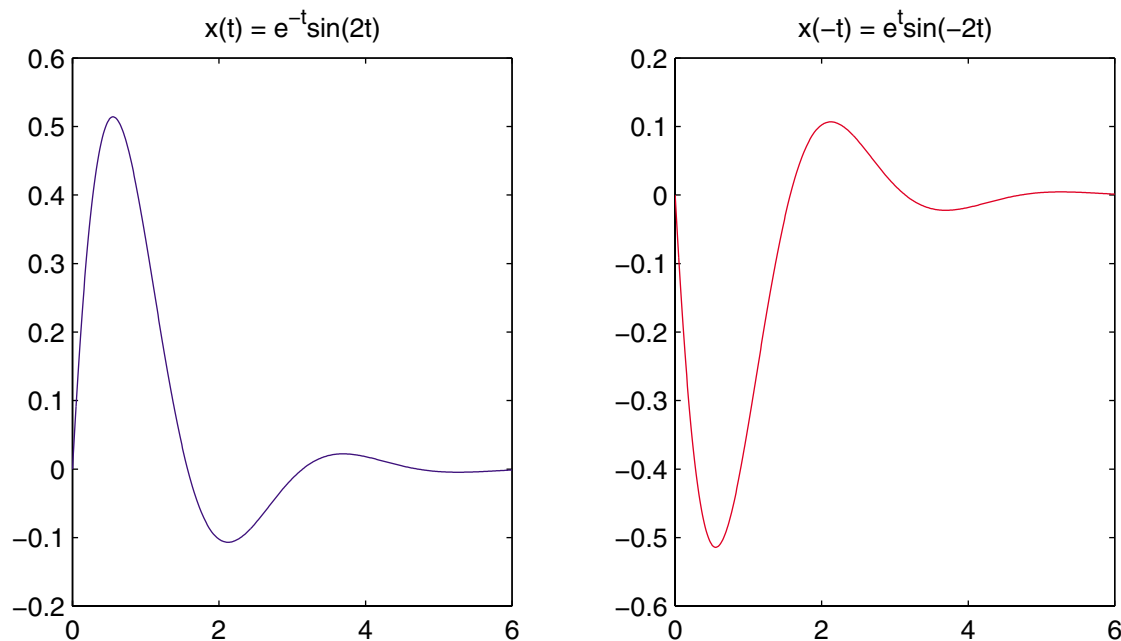
Time Reversal

If $x(t)$ is a continuous function, the time-reversed signal is defined as $y(t) = x(-t)$.

Recall that an even signal is the same as its reflected version, i.e., $x(-t) = x(t)$ and an odd signal is the negative of its reflected version, i.e., $x(-t) = -x(t)$.

Example 1.10 (chs1_ex10.m)

Consider $x(t) = e^{-t} \sin 2t$, This signal and its time-reversed $x(-t)$ is as shown.



Time Scaling

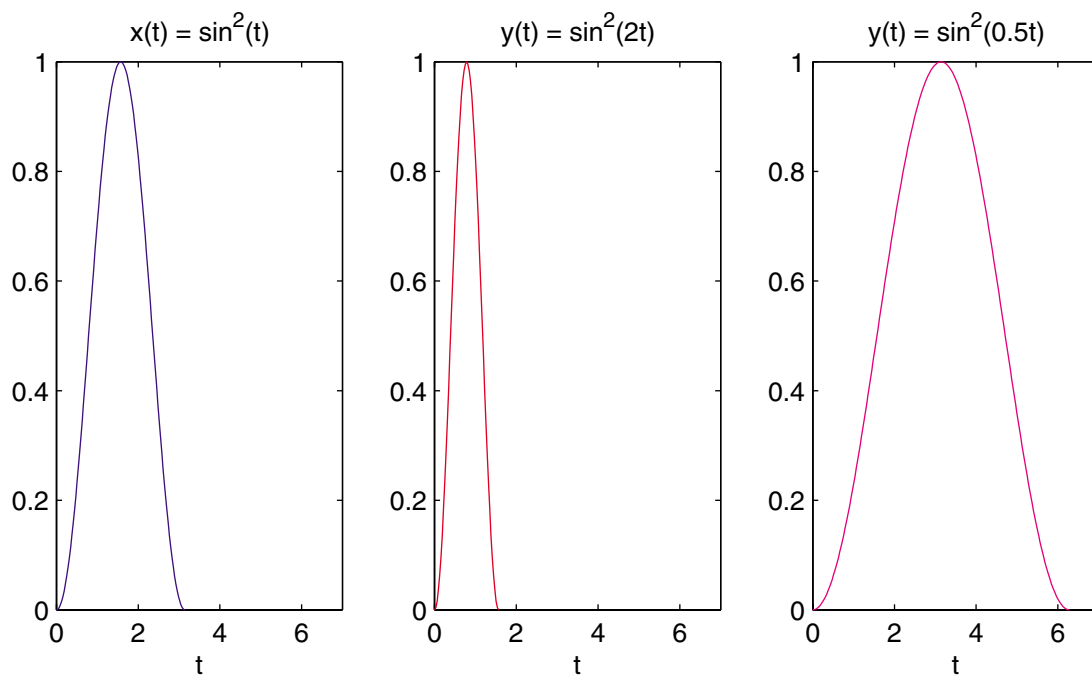
If $x(t)$ is a continuous function, a time-scale version of this signal is defined as $y(t) = x(at)$.

If $a > 1$, the signal $y(t)$ is a compressed version of $x(t)$, i.e., the time interval is compressed to $\frac{1}{a}$.

If $0 < a < 1$, the signal $y(t)$ is a stretched version of $x(t)$, i.e., the time interval is stretched by $\frac{1}{a}$.

Example 1.11 (chs1_ex11.m)

Consider $x(t) = \sin^2 t$, This signal and its time-scaled $y(t) = \sin^2 2t$, and $\sin^2 \frac{1}{2}t$ over one-half period is as shown.



Composite Time transformation

If $x(t)$ is a continuous signal, the scale and shifting transformation of the signal is given by $y(t) = x(at + t_0)$, if $a < 0$, the signal will go through a reversal as well. The general method to obtain a time transformation is as follows:

1. Given $x(t)$, replace t with τ
2. Given the transformation $\tau = at + t_0$, solve for t
3. Draw the t -axis directly below the τ -axis
4. Plot $y(t)$ on the t -axis

Amplitude Transformation

The transformation in signal amplitude involves

Amplitude scaling

If $x(t)$ is a continuous signal, the amplitude transformation of this signal is

$$y(t) = ax(t)$$

Addition

If $x_1(t)$, and $x_2(t)$ are two continuous signals, the addition of the signals is

$$y(t) = x_1(t) + x_2(t)$$

Multiplication

If $x_1(t)$, and $x_2(t)$ are two continuous signals, the product of the signal is

$$y(t) = x_1(t)x_2(t)$$

We have similar time and amplitude transformations for discrete time signals.

Sinusoidal Signals

The sinusoidal signal is of major importance since if we know the response of a system to sinusoids of all frequencies, we have a complete description of the system. The continuous-time version of this signal is written as

$$x(t) = A \cos(\omega_0 t + \theta)$$

where A is the amplitude, ω_0 is the frequency in radians per second, and θ is the phase angle in radians. The period of the signal is $T = \frac{2\pi}{\omega_0}$. The manipulation and analysis of sinusoidal signals is simplified by dealing with their amplitude and phase angles or their phasor representation.

From Euler's identity, the complex exponential function is expressed as

$$e^{j\beta} = \cos \beta + j \sin \beta \quad \text{or}$$

$$\cos \beta = \Re e^{j\beta} \quad \text{and} \quad \sin \beta = \Im e^{j\beta}$$

Thus the sinusoidal signal $x(t)$ may be written as

$$x(t) = \Re\{Ae^{j(\omega_0 t + \theta)}\} = \Re\{Ae^{j\theta}e^{j\omega_0 t}\} = \Re\{\vec{X}e^{j\omega_0 t}\}$$

Note that $Ae^{j\theta}$ indicates the magnitude and phase angle of the sinusoidal signal $x(t)$. This complex number is called *phasor* and is shown by $\vec{X} = Ae^{j\theta}$, or $\vec{X} = A\angle\theta$. We say that we transform a sinusoidal signal from time domain into phasor domain, and we write

$$\begin{aligned}\mathcal{P}\{A \cos(\omega_0 t + \theta)\} &= Ae^{j\theta} \\ &= A\angle\theta\end{aligned}$$

We define the complex signal as

$$\tilde{x}(t) = Ae^{j(\omega_0 t + \theta)} = \vec{X}e^{j\omega_0 t}$$

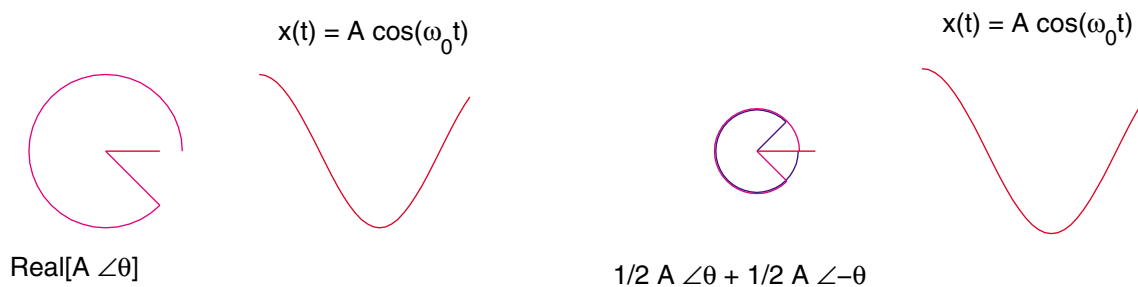
\vec{X} is a rotating vector or a phasor whose real part generates the cosine signal $x(t)$. Rewriting the complex signal $\tilde{x}(t)$ and its conjugate as

$$\begin{aligned}\tilde{x}(t) &= Ae^{j(\omega_0 t + \theta)} = A[\cos(\omega_0 t + \theta) + j \sin(\omega_0 t + \theta)] \\ \tilde{x}^*(t) &= Ae^{-j(\omega_0 t + \theta)} = A[\cos(\omega_0 t + \theta) - j \sin(\omega_0 t + \theta)]\end{aligned}$$

We conclude that the cosine signal $x(t)$ can be expressed as one-half of the sum of the complex signal $\tilde{x}(t)$ with positive frequency ω_0 and its conjugate $\tilde{x}^*(t)$, with negative frequency $-\omega_0$, i.e.,

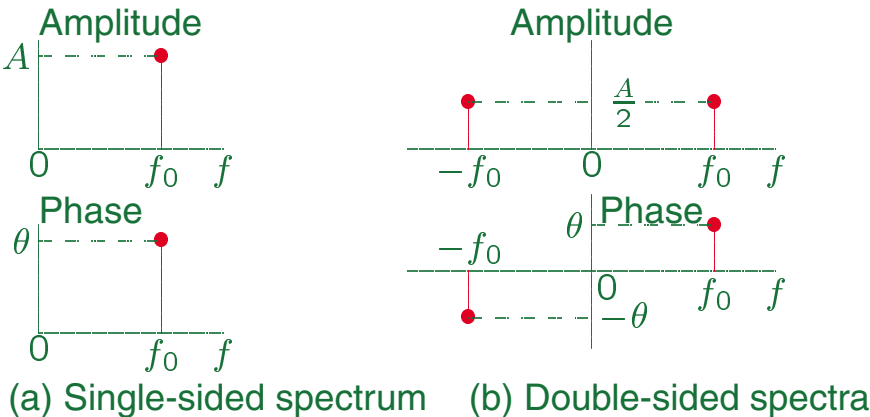
$$x(t) = \frac{1}{2} [(\tilde{x}(t) + \tilde{x}^*(t))]$$

Therefore, the oppositely rotating phasors $\frac{1}{2}\vec{X} = \frac{1}{2}A\angle\theta$ and $\frac{1}{2}\vec{X}^* = \frac{1}{2}A\angle-\theta$ will generate the cosine signal $x(t)$. The script file **rotphasor** illustrates how the real part of a rotating phasor generates a cosine signal, and the file **rot2phasor** demonstrates how the sum of two half-amplitude oppositely rotating phasors generates the cosine signal. To see a demonstration type **rotphasor**, and **rot2phasor** at the MATLAB prompt.



Frequency Domain Spectra

An alternative way to visualize the sinusoidal signal $x(t)$ in the frequency domain is in the form of two plots. One the amplitude A as the function of frequency f , and the other its phase angle θ as a function of f . These plots are referred to as *single-sided spectrum*. If the amplitude and phase angle plots are made for the oppositely rotating phasors we obtain the so called double-sided spectra as shown.



Note that if a signal is represented as a sine function, before finding the signal spectra it must be expressed in terms of a cosine function, $\sin(\omega_0 t + \theta) = \cos(\omega_0 t + \theta - \frac{\pi}{2})$.

Example 1.12

For the signal

$$x(t) = 6 \cos\left(10\pi t + \frac{\pi}{6}\right) + 4 \sin\left(18\pi t + \frac{\pi}{6}\right)$$

- (a) Find the period and the fundamental frequency of the signal.
- (b) Write the signal as the real part of the sum of the rotating phasors.
- (c) Write $x(t)$ as the sum of counter rotating phasors.
- (d) plot the single-sided amplitude and phase spectra.
- (e) plot the 2-sided amplitude and phase spectra.

(a) Converting the second term to a cosine function, we have

$$x(t) = 6 \cos\left(10\pi t + \frac{\pi}{6}\right) + 4 \cos\left(18\pi t - \frac{\pi}{3}\right)$$

We have $10\pi = 2\pi m f_0$ and $18\pi = 2\pi n f_0$, where, m , and n are integers and f_0 is the largest constant that satisfies these equations. Therefore,

$$f_0 = \frac{10\pi}{2\pi m} = \frac{18\pi}{2\pi n} \quad \Rightarrow \quad \frac{5}{m} = \frac{9}{n}$$

or

$$n = 1.8m$$

for $m = 5$ $n = 9$, therefore

$$f_0 = \frac{10}{(2)(5)} = 1, \quad \text{and} \quad T_0 = 1.0\text{s}$$

(b) $x(t)$ in terms of the real part of the rotating phasors is

$$x(t) = \Re \left[6e^{j(10\pi t + \frac{\pi}{6})} \right] + \Re \left[4e^{j(18\pi t - \frac{\pi}{3})} \right]$$

(c) $x(t)$ in terms of the of counter rotating phasors is

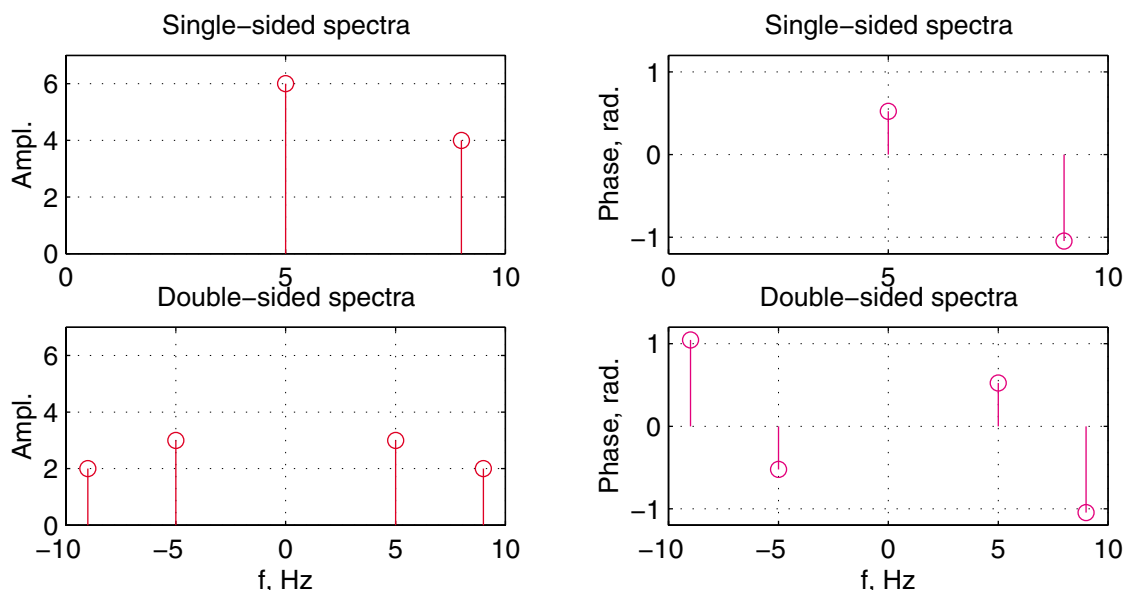
$$\begin{aligned} x(t) = & \left[3e^{j(10\pi t + \frac{\pi}{6})} \right] + \left[3e^{-j(10\pi t + \frac{\pi}{6})} \right] \\ & + \left[2e^{j(18\pi t - \frac{\pi}{3})} \right] + \left[2e^{-j(18\pi t - \frac{\pi}{3})} \right] \end{aligned}$$

(d)-(e) The single-sided amplitude spectra are 6 at $f_0 = 5$ Hz, and 4 at $f_0 = 9$ and the corresponding phase angles are $\frac{\pi}{6}$, and $-\frac{\pi}{3}$. The double-sided amplitude spectra is given by one-half the above amplitudes their mirror image. The phase spectra is obtained by taking antisymmetric mirror image.

The following MATLAB commands are used to plot the single-sided and double-sided spectra.

```
%chs1_ex12.m
f = [5, 9]; % Fundamental frequencies
A = [6, 4]; % Corresponding amplitudes
theta = [pi/6, -pi/3]; % corresponding angles
subplot(2,2,1),stem(f, A, 'r') % Single-sided amplitude
subplot(2,2,2),stem(f,theta,'m') % and angle spectra
f = [-f f]; % -ve and +ve frequencies
AD = [A/2 A/2]; % 1/2 A mirror image, and 1/2 A
thetaD = [-theta theta]; %antisymmetric angle, and angle
subplot(2,2,3), stem(f, AD,'r') % double-sided amplitude
subplot(2,2, 4), stem(f, thetaD, 'm')% and angle spectra
```

The result is

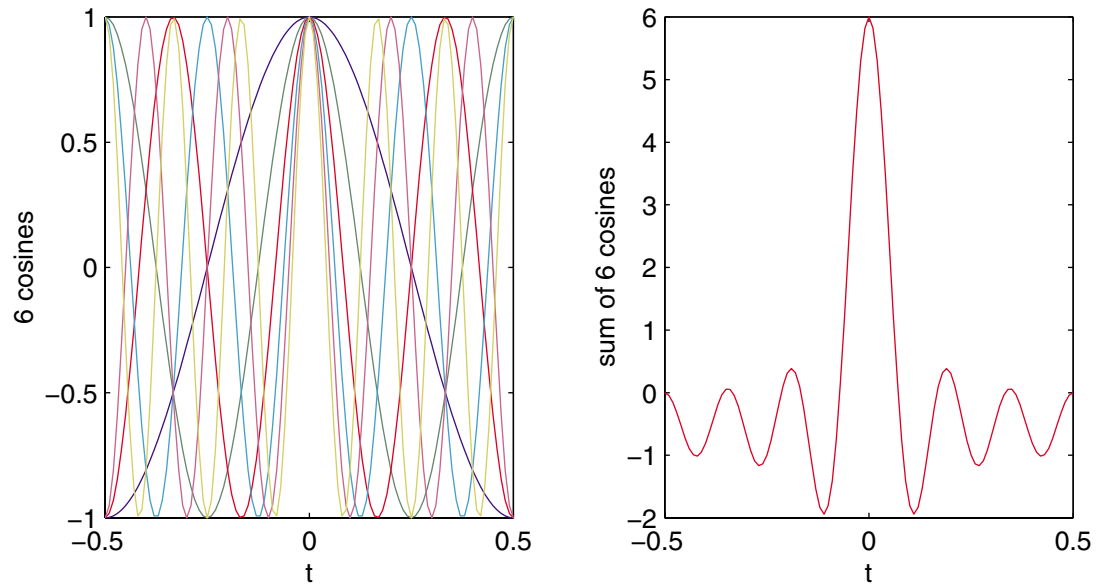


Special signals

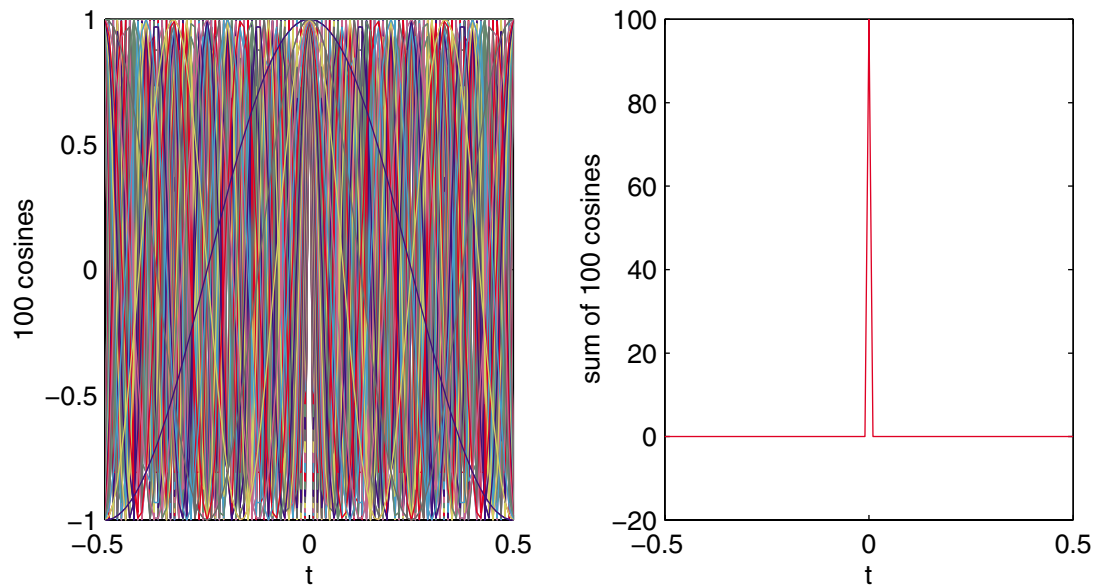
Consider the cosine function $\cos k(2\pi t)$. What function results from the sum of signals of all frequencies, i.e., $\sum_{k=1}^{\infty} \cos k(2\pi t)$ over one period? To see this, we write the following commands to obtain a plot of the individual signals and their sum.

```
%sumcosine.m
close          % Close the figure
t = -0.5:.01:0.5; % Time interval
m =length(t);
n = input('No. of sequences -> ');
x = zeros(n, m); % Initialize x
sumx = zeros(1,m); % Initialize sum of cosines
for k = 1:n;
    tn= k*t;
    x(k,:)= cos(2*pi*tn);
    sumx = sumx + x(k,:);
end
subplot(1,2,1), plot(t, x, '-'),
xlabel('t'), ylabel([num2str(n), ' cosines'])
subplot(1,2,2), plot(t, sumx, 'r'), hold on;
xlabel('t'),ylabel(['sum of ',num2str(n),' cosines'])
```

The result for $n = 6$ is as shown.



The results for $n = 100$ is.



Since all cosines equal unity at $t = 0$, as $n \rightarrow \infty$, the resultant amplitude approaches infinity at origin, and zero elsewhere. This signal with the amplitude approaching infinity and duration to zero (Subject to unit area) is called a unit *impulse*. The graphic symbol for the impulse function is an arrow.

The steady-state response of a system to a sinusoidal input is known as the frequency response. Thus, we can find the frequency response of the system over the entire frequency range by applying a single impulse as the input. We shall learn that the response of a system to a complicated signal can be obtained by expressing the input as a sum of weighted impulses. (convolution). The impulse function, together with the unit step function are very important in signal processing and their properties are discussed next.

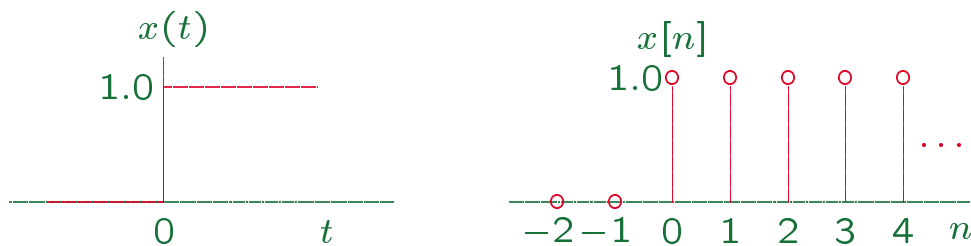
Unit Step Function

The continuous-time unit step function denoted by $u(t)$, is defined by

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

The discrete-time step function, denoted by $u[n]$, is defined by

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

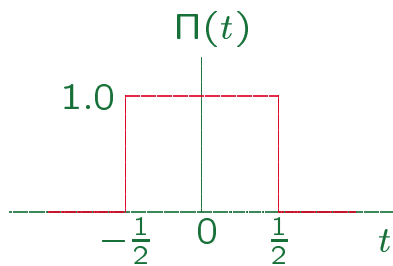


The step input $u(t)$ represent an abrupt change in the signal, for example as a result of a dc voltage applied at $t = 0$ by closing a switch. The step input is important because the step response of a system displays important characteristics of the system.

Unit Pulse Function

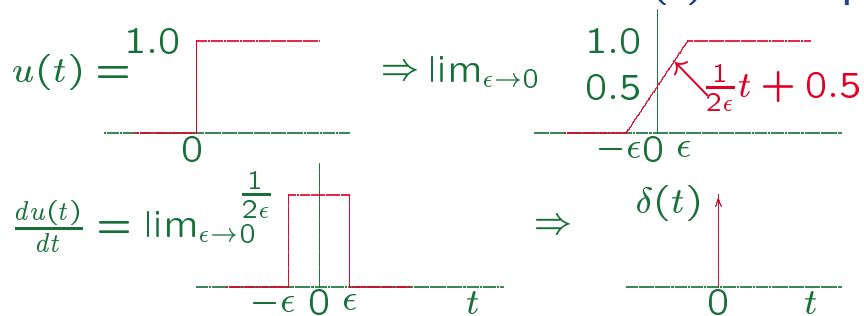
The continuous-time unit impulse function denoted by $\Pi(t)$, is defined by

$$\Pi(t) = \begin{cases} 1, & -\frac{1}{2} \leq t \leq \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$



Unit Impulse Function

The unit step function shown in (a) has discontinuity at $t = 0$. We like to find the derivative of $u(t)$ at the point of discontinuity.

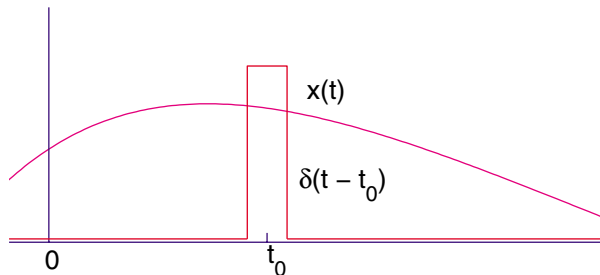


Assume that $u(t)$ varies linearly from $-\epsilon$ to ϵ as shown in (b). In the limit as $\epsilon \rightarrow 0$, we have the original step function. The derivative of (b) is shown in (c). We observe that as $\epsilon \rightarrow 0$ the amplitude of $u'(t)$ approaches infinity, and the duration approaches to zero. Furthermore the area under $u'(t)$ between $\pm\epsilon$ remain constant at unity. We say that the function between $\pm\epsilon$ approaches a *unit impulse function*, denoted by $\delta(t)$. One way to visualize $\delta(t)$ is to view it as the limiting form of a rectangular pulse of unit area. Mathematically this is described as

$$\delta(t) = 0 \quad \text{for } t \neq 0 \quad \text{and}$$

$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$

Sifting property of an impulse



Consider a continuous-time signal $x(t)$ multiplied by $\delta(t - t_0)$ for some fixed t_0 . Integrating over the entire real line, and in the light of the unit impulse definition, the integral of this product is

$$\int_{-\infty}^{\infty} x(t)\delta(t - t_0)dt = x(t_0)$$

That is, the operation on the left hand side sifts out the value $x(t_0)$ of the function $x(t)$ at $t = t_0$. This extremely important result is called the *sifting property*. This property is regarded as the formal definition of an impulse.

We noticed that that $\delta(t)$ is the limiting form of a rectangular pulse of unit area as $\epsilon \rightarrow 0$, thus, it follows that the impulse δ is an even function, i.e.,

$$\delta(-t) = \delta(t)$$

The next property of an impulse is the scaling property. Viewing the impulse as a limiting form a rectangular pulse of unity, it follows that the time compressed impulse $\delta(at)$ has an area of $\frac{1}{|a|}$, thus

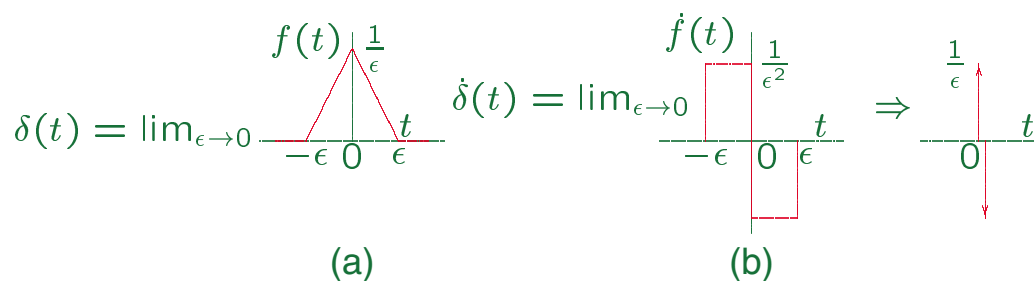
$$\delta(at) = \frac{1}{|a|}\delta(t)$$

Another important property of impulse is the product property. Again let us view the impulse as the limiting form of a tall narrow rectangular pulse of unit area occurring at $t = t_0$. The product of a signal $x(t)$ with this pulse is also a tall, narrow pulse whose height is scaled by the value of $x(t)$ at $t = t_0$, or $x(t_0)$, therefore, we have

$$x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0) \quad \& \quad x(t)\delta(t) = x(0)\delta(t)$$

The Doublet (Derivative of an impulse)

The limiting form of the triangular signal shown in Figure (a) generates an impulse as $\epsilon \rightarrow 0$. The limiting form of the signal shown in Figure (b) as $\epsilon \rightarrow 0$ represent the derivative of the impulse. This signal is referred to as a moment or a doublet signal.



The doublet signal is an odd signal

The Doublet Properties

The doublet signal has similar properties as impulse signal. These properties such as sifting, product, and scaling are summarized below

$$\int_{-\infty}^{\infty} x(t)\dot{\delta}(t - t_0) dt = -\dot{x}(t)$$
$$x(t)\dot{\delta}(t - t_0) = x(t_0)\dot{\delta}(t - t_0) - \dot{x}(t_0)\delta(t - t_0)$$
$$\dot{\delta}(-t) = -\dot{\delta}t$$

Energy and Power Signals

Consider a resistor R supplied by a voltage $v(t)$, the instantaneous power dissipated in R is

$$p(t) = Ri^2(t) = \frac{v^2(t)}{R}$$

The instantaneous Power on a per ohm basis can be written as

$$p(t) = x^2(t)$$

where $x(t)$ represent a voltage or current. Since power is the rate of energy, we define the energy in the interval $-T < t < T$ as

$$\begin{aligned} E &= \lim_{T \rightarrow \infty} \int_{-T}^T |x^2(t)| dt \\ &= \int_{-\infty}^{\infty} |x^2(t)| dt \end{aligned}$$

and the average power normalized to unit resistance is

$$P = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |x^2(t)| dt$$

For a periodic signal $x(t)$ with the fundamental period T , the average power is

$$P = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x^2(t)| dt$$

The square root of the average power P is called the *root-mean-square* (rms) value of the signal $x(t)$. For the DT signal $x[n]$, the integrals are replaced by sums, and the signal energy is

$$E = \sum_{-N}^N x^2[n]$$

and the average power is

$$P = \lim_{N \rightarrow \infty} \frac{1}{2N} \sum_{-N}^N x^2[n]$$

For a periodic signal $x[n]$, with fundamental period N , the average power is

$$P = \frac{1}{N} \sum_0^{N-1} x^2[n]$$

A signal is referred to as *energy signal*, if and only if the total energy of the signal satisfies the condition

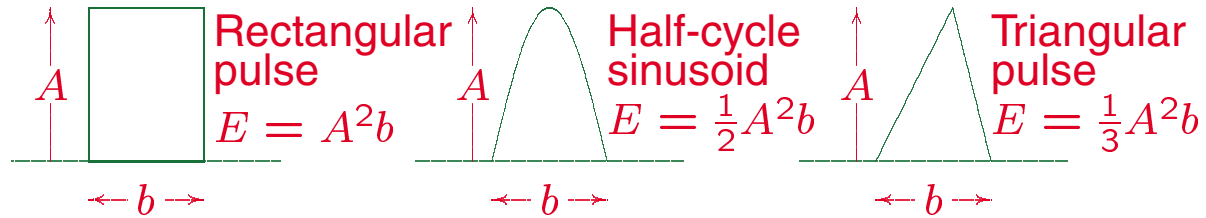
$$0 < E < \infty \quad \text{so that } P = 0$$

A signal is referred to as *power signal*, if and only if the total energy of the signal satisfies the condition

$$0 < P < \infty \quad \Rightarrow E = \infty$$

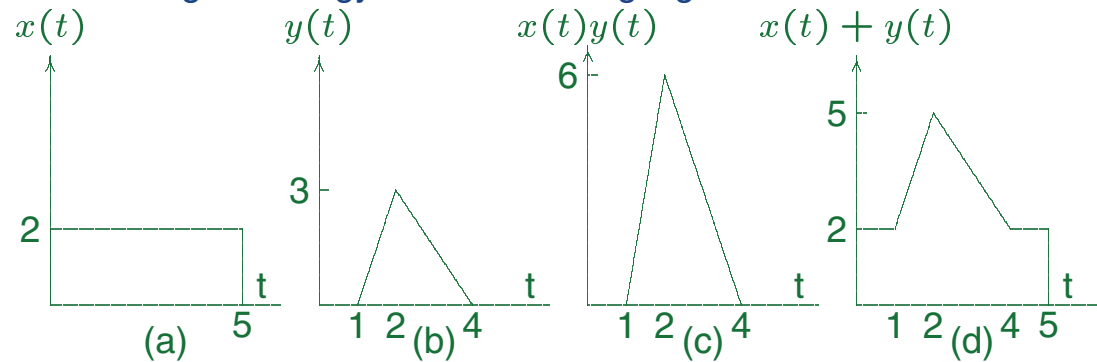
An energy signal has zero average power, whereas a power signal has infinite energy. Thus the periodic signals are classified as power signals.

The signal energy of three typical pulse shown are



Example 1.14

Find the signal energy for the following signals



(a) $E = (2)^2(5) = 20$

(b) $E = \frac{1}{3}(3)^2(3) = 9$

(c) $E = \frac{1}{3}(6)^2(3) = 36$

(d) $E = \int_{-\infty}^{\infty} [x(t) + y(t)]^2 dt$
 $= \int_{-\infty}^{\infty} [x^2(t) + y^2(t) + 2x(t)y(t)] dt$
 $= E_x + E_y + 2 \int_{-\infty}^{\infty} x(t)y(t) dt$
 $= 20 + 9 + 72 = 101$

Example 1.15

Find the total energy and the average power of the periodic signal $x(t) = \sin t$

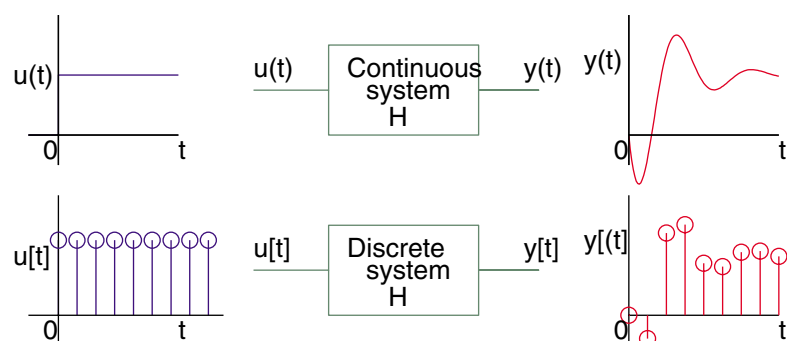
$$\begin{aligned} E &= \int_{-\infty}^{\infty} \sin^2(t) dt \\ &= \int_{-\infty}^{\infty} \frac{1}{2} (1 - \cos 2t) dt \\ &= \frac{1}{2} \left(t - \frac{1}{2} \sin 2t \right) \Big|_{-\infty}^{\infty} = \infty \end{aligned}$$

The average power is

$$\begin{aligned} P &= \frac{1}{2T} \int_{-T}^T \sin^2(t) dt \\ &= \frac{1}{2T} \int_{-T}^T \frac{1}{2} (1 - \cos 2t) dt \\ &= \frac{1}{4T} \left(t - \frac{1}{2} \sin 2t \right) \Big|_{-T}^T = \frac{1}{2} \end{aligned}$$

Systems

A system is an interconnection of components that transforms an input signal into an output signal. We can therefore view a system as a **mapping** from an input function onto an output function.



continuous-time systems Input and output are continuous-time signals, and we write

$$y(t) = \mathcal{H}u(t)$$

Discrete-time system Input and output are discrete-time signals, and we write

$$y[n] = \mathcal{H}u[n]$$

Hybrid system may have CT and DT input and output signals.

Instantaneous Systems

Some systems may be modeled by instantaneous (Non-dynamic) linear relationships such as

$$y(t) = Ax(t) + B$$

where A and B are constants, or a nonlinear relationship such as

$$y(t) = Ax^2(t) + Bx(t) + C$$

Instantaneous systems are referred to as *memoryless* since the output $y(t)$ depends only on the instantaneous value of $x(t)$. For example, a resistor is memoryless since in cause and effect relation $i(t) = \frac{1}{R}v(t)$, the current at time t depends on the the input at time t . The discrete-time signal described by

$$y[n] = 2x[n]$$

is memoryless, since the value of $y[n]$ at time n depends only on the present value of the input $x[n]$.

Dynamic Systems

Continuous-time systems are often modeled by linear time-invariant differential equation. In general the input-output relationship may be given by

$$\frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \dots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_m \frac{d^m x(t)}{dt^m} + b_{m-1} \frac{d^{m-1} x(t)}{dt^{m-1}} + \dots + b_1 \frac{dx(t)}{dt} + b_0 x(t)$$

The coefficients of the above differential equation are the parameters of the physical system. Dynamic systems are systems with memory. For example an inductor has memory, since the current in an inductor given by

$$i(t) = \frac{1}{L} \int_{-\infty}^t v(\tau) d\tau$$

depends on all past values of the voltage $v(t)$. The memory of an inductor extends into the infinite past.

Time-invariant and Time-varying Systems

A system is **time-invariant** or **fixed**, if its input-output relationship does not change with time, i.e., if $\mathcal{H}[x(t)] = y(t)$, then

$$\mathcal{H}[x(t - \tau)] = y(t - \tau)$$

For example, the system described by

$$y(t) = x(t) + Ax(t - T)$$

is time-invariant if A and T are constants. Also, if the coefficients of the differential equation described earlier are constant, the system is referred to as a time-invariant system. In the time-invariant systems the shape of the response $y(t)$ depends only on the shape of the input $x(t)$ and not on the time when it is applied. When one or more coefficients are function of time, the system is called **time-varying system**.

Causal and Noncausal Systems

A system is *causal* or *non-anticipating* if its present response depends on the present and/or past values of the input signal. Systems whose present response is affected by future inputs are termed noncausal or anticipating. All physical systems are causal, however ideal systems such as ideal filters often turn out to be noncausal. For example, the average system described by

$$y[n] = \frac{1}{3} (x[n - 2] + x[n - 1] + x[n])$$

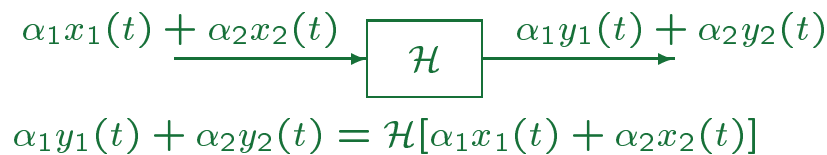
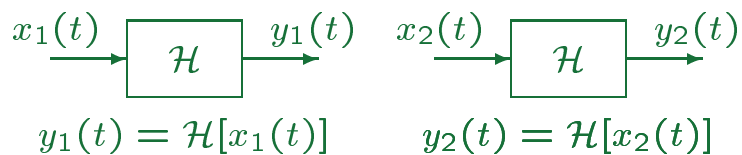
is causal, whereas, the average system described by

$$y[n] = \frac{1}{3} (x[n - 1] + x[n] + x[n + 1])$$

is noncausal, since the output signal $y[n]$ depends on a future value of the input signal $x[n + 1]$.

Linear and Nonlinear Systems

A system is linear if it satisfies the *principle of superposition*



That is, the response of a linear system to a weighted sum of inputs equal to the same weighted sum of output signals.

Example 2.1

The system described by

$$\frac{dy}{dt} + ty(t) = x(t)$$

is linear (but time-varying), since for signals $x_1(t)$, and $x_2(t)$

$$\frac{dy_1}{dt} + ty_1(t) = x_1(t) \quad \text{and}$$

$$\frac{dy_2}{dt} + ty_2(t) = x_2(t) \quad \text{then}$$

$$\alpha_1 \frac{dy_1}{dt} + \alpha_2 \frac{dy_2}{dt} + \alpha_1 ty_1(t) + \alpha_2 ty_2(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t)$$

$$\frac{d}{dt}(\alpha_1 y_1(t) + \alpha_2 y_2(t)) + t(\alpha_1 y_1(t) + \alpha_2 y_2(t)) = \alpha_1 x_1 + \alpha_2 x_2$$

That is, the response to the input $\alpha_1 x_1(t) + \alpha_2 x_2(t)$ is $\alpha_1 y_1(t) + \alpha_2 y_2(t)$. But the system is time varying since the coefficient of $y(t)$ is not a constant.

Example 2.2

Is the system described by $y[n] = 3u[n] + 5$ linear or nonlinear?

Let $u_1[n] = 1$, and $u_2[n] = 2$, then

$$y_1[n] = 3u_1[n] + 5 = 8$$

$$y_2[n] = 3u_2[n] + 5 = 11$$

The sum of two response is 19, but the sum of the input signals yield the response

$$y_3[n] = 3(u_1[n] + u_2[n]) + 5 = 14$$

which is not the same. therefore, the system is non-linear.

In linear systems, scaling input leads to an ideal scaling of the output. This means zero output for zero input and a linear input-output relation passing through origin.

A system with initial condition can be treated as a multiple-input system with initial condition acting alone as an additional input.

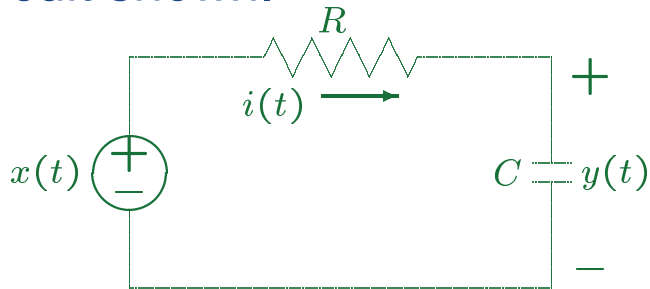
Nonlinear Systems

A system is nonlinear if it has

- Nonlinear elements
- Nonzero initial conditions
- Internal sources

Convolution

The concept of convolution is first introduced by finding the solution of a first order differential equation by the techniques of the variation of parameters. An example of a linear time-invariant system leading to a first order differential equation is the familiar RC circuit shown.



The output $y(t)$ is the voltage across the capacitor with the initial value of y_0 at time $t = t_0$.

Applying KVL, we have

$$x(t) = Ri(t) + y(t) \quad \text{and} \quad i(t) = C \frac{dy(t)}{dt}$$

Substituting for i

$$RC \frac{dy(t)}{dt} + y(t) = x(t)$$

RC is the circuit time constant shown by τ , thus

$$\tau \frac{dy(t)}{dt} + y(t) = x(t)$$

This is a first-order differential equation. The complete solution is the sum of the homogeneous solution (natural response or zero-input response), and the particular integral solution (force response or the zero state response). For the homogeneous solution, we have

$$\tau \frac{dy_h(t)}{dt} + y_h(t) = 0$$

Assuming a solution of the form $y_h(t) = Ae^{pt}$, and substituting into the homogeneous equation, we find $p = -\frac{1}{\tau}$, and the homogeneous solution

$$y_h(t) = Ae^{-\frac{t}{\tau}}$$

The total solution of the nonhomogeneous equation

$$\tau \frac{dy(t)}{dt} + y(t) = x(t)$$

can be obtained by applying the technique of *variation of parameters*, which is the same form of solution as $y_h(t)$, except the coefficient A is a function of time, shown by $A(t)$,

$$y(t) = A(t)e^{-\frac{t}{\tau}}$$

Differentiating by means of chain rule, we have

$$\frac{dy(t)}{dt} = \left[\frac{dA(t)}{dt} - \frac{A(t)}{\tau} \right] e^{-\frac{t}{\tau}}$$

Substituting for $y(t)$ and $\frac{dy(t)}{dt}$ in the nonhomogeneous equation, we get

$$\tau \left[\frac{dA(t)}{dt} - \frac{A(t)}{\tau} \right] e^{-\frac{t}{\tau}} + A(t)e^{-\frac{t}{\tau}} = x(t)$$

or

$$\frac{dA(t)}{dt} = \frac{1}{\tau} x(t) e^{\frac{t}{\tau}}$$

Solving for $A(t)$, and using λ as a dummy variable, we obtain

$$A(t) = \int_{t_0}^t x(\lambda) \frac{1}{\tau} e^{\frac{\lambda}{\tau}} d\lambda + A(t_0)$$

where $A(t_0) = y_0 e^{\frac{t_0}{\tau}}$. Substituting for $A(t)$ in the assumed solution, results in

$$\begin{aligned} y(t) &= \left[\int_{t_0}^t x(\lambda) \frac{1}{\tau} e^{\frac{\lambda}{\tau}} d\lambda + y_0 e^{\frac{t_0}{\tau}} \right] e^{-\frac{t}{\tau}} \\ &= y_0 e^{-\frac{t-t_0}{\tau}} + \int_{t_0}^t x(\lambda) \frac{1}{\tau} e^{-\frac{(t-\lambda)}{\tau}} d\lambda \end{aligned}$$

If the input $x(t)$ is applied at $t = -\infty$, and $y_0 = y(-\infty) = 0$, we have

$$y(t) = \int_{-\infty}^t x(\lambda) \frac{1}{\tau} e^{-\frac{t-\lambda}{\tau}} d\lambda$$

If we define

$$h(t) = \frac{1}{\tau} e^{-\frac{t}{\tau}} u(t)$$

then

$$h(t - \lambda) = \frac{1}{\tau} e^{-\frac{(t-\lambda)}{\tau}} u(t - \lambda)$$

The solution is

$$y(t) = \int_{t_0}^t x(\lambda)h(t - \lambda)d\lambda$$

or

$$y(t) = \int_{t_0}^t h(\lambda)x(t - \lambda)d\lambda$$

Recall that we showed $\frac{1}{\tau}e^{-\frac{t}{\tau}}u(t)$ by $h(t)$. We can show that this is the impulse response of the RC circuit described by (Example 2-11 Textbook)

$$\tau \frac{dy(t)}{dt} + y(t) = x(t) \quad -\infty < t < \infty$$

For an impulse input $x(t) = \delta(t)$, the response $y(t) = h(t)$, and since for $t > 0$, $\delta(t) = 0$, the governing differential equation for the impulse response is

$$\tau \frac{dh(t)}{dt} + h(t) = 0 \quad t > 0$$

Assuming a solution of the form $h(t) = Ae^{pt}$ and substituting in the given equation, we obtain

$$(\tau p + 1)Ae^{pt} = 0 \quad \Rightarrow \quad p = \frac{-1}{\tau}$$

Therefore,

$$h(t) = Ae^{\frac{-t}{\tau}} \quad t > 0$$

To find A , we need $h(0^+)$, so rewriting the given equation for $t \geq 0$

$$\tau \frac{dh(t)}{dt} + h(t) = \delta(t) \quad t \geq 0$$

In the above relation for the left hand side to be identically equal to the right hand side the term in the left hand side must be an impulse. $h(t)$ cannot be an impulse since $\tau \frac{dh(t)}{dt}$ must be a unit doublet. Therefore, $\int_{0^-}^{0^+} h(t)dt = 0$. The integral from $t = 0^-$ to $t = 0^+$ is

$$\int_{t=0^-}^{0^+} \tau \frac{dh(t)}{dt} dt + \int_{t=0^-}^{0^+} h(t) dt = \int_{t=0^-}^{0^+} \delta(t) dt$$
$$\tau [h(0^+) - h(0^-)] + 0 = 1$$

Since $h(0^-) = 0$, we have $h(0^+) = \frac{1}{\tau}$, therefore $A = \frac{1}{\tau}$, and the impulse response is

$$h(t) = \frac{1}{\tau} e^{-\frac{t}{\tau}}$$

Rewriting this important result again, the response of a continuous time-invariant system $y(t)$, to an input $x(t)$ is given by

$$y(t) = \int_{t_0}^t x(\lambda) h(t - \lambda) d\lambda$$

or

$$y(t) = \int_{t_0}^t h(\lambda) x(t - \lambda) d\lambda$$

where $h(t)$ is the impulse response of the system. This important result is known as the *convolution integral*. The above convolution integral are often written symbolically as

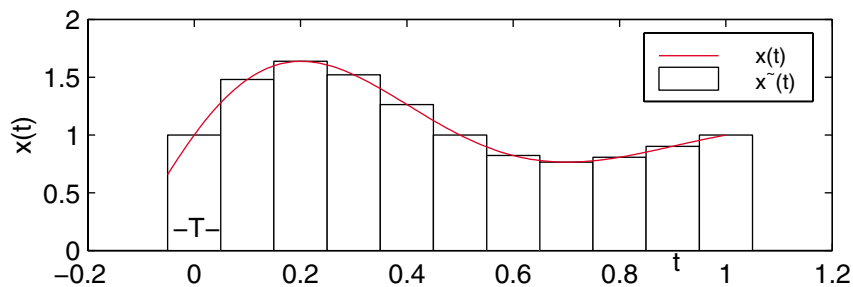
$$y(t) = h(t) * x(t)$$

$$y(t) = x(t) * h(t)$$

Superposition Integral for LTI Systems

The response of a LTI system to an input signal $x(t)$ can be obtained by expressing the input $x(t)$ as a weighted superposition of time-shifted impulses. We divide $x(t)$ into narrow rectangular strips of width T , at nT , $n = \pm 1, \pm 2, \dots$, and replace each strip by an impulse with strength $T \cdot x(nT)$ which is the area under each strip.

$$\tilde{x}(t) = \sum_{n=-\infty}^{\infty} T \cdot x(nT) \delta(t - nT) \quad \text{Sum of shifted impulses}$$



By superposition the response is a weighted superposition of time-shifted responses, i.e.,

$$y(t) = \sum_{n=-\infty}^{\infty} T \cdot x(nT)h(t-nT) \quad \text{Sum of shifted impulse responses}$$

In the limit as $T \rightarrow d\lambda \rightarrow 0$ with nT as a continuous variable λ

$$\begin{aligned} x(t) &= \lim_{T \rightarrow d\lambda} \tilde{x}(t) = \lim_{T \rightarrow d\lambda} \left[\sum_{n=-\infty}^{\infty} T \cdot x(nT) \delta(t - nT) \right] \\ &= \int_{-\infty}^{\infty} x(\lambda) \delta(t - \lambda) d\lambda \end{aligned}$$

and

$$\begin{aligned} y(t) &= \lim_{T \rightarrow d\lambda} \left[\sum_{n=-\infty}^{\infty} T \cdot x(nT)h(t - nT) \right] \\ &= \int_{-\infty}^{\infty} x(\lambda)h(t - \lambda) d\lambda \end{aligned}$$

For casual systems the upper limit can be set to t , also, if $x(t) = 0$ for $t < 0$, the lower limit becomes zero and we have

$$y(t) = \int_0^t x(\lambda)h(t - \lambda)d\lambda$$

Alternatively, making a change of variable $\sigma = t - \lambda$, we have

$$y(t) = \int_0^t x(t - \sigma)h(\sigma)d\sigma$$

In summary, the response of a LTI system to an input $x(t)$ is expressed as a *Convolution integral*

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\lambda)h(t - \lambda)d\lambda$$

where $h(t)$ is the impulse response. We can view this integral as the product of the input and a shifted and flipped version of the impulse response and then evaluating the area of the product $x(\lambda)h(t - \lambda)$ as t increases. Alternatively, we can write

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\lambda)x(t - \lambda)d\lambda$$

Some Properties of Convolution

1. $h(t) * x(t) = x(t) * h(t)$
2. $h(t) * [\alpha x(t)] = \alpha [h(t) * x(t)]$
3. $h(t) * [x_1(t) + x_2(t)] = h(t) * x_1(t) + h(t) * x_2(t)$
4. $h(t) * [x_1(t) * x_2(t)] = [h(t) * x_1(t)] * x_2(t)$
5. The starting time of $y(t)$ equals the sum of the starting times of $x(t)$ and $h(t)$. The ending time of $y(t)$ equals the sum of the ending times of $x(t)$ and $h(t)$
6. The duration of $y(t)$ equals the sum of the durations of $x(t)$ and $h(t)$
7. The value of $y(t)$ equals the area of the product of $x(t)$ and $h(t)$

Graphical Convolution

In some cases the convolution integral may be obtained by evaluating the area formed by the product $h(\lambda)$ and $x(t - \lambda)$, or $x(\lambda)$ and $h(t - \lambda)$. The choice of which function to fold and slide is arbitrary. With the first choice, we shift $x(t - \lambda)$ from a position of no overlap with $x(t)$ over the entire ranges of overlap. We establish the correct ranges as follows:

- set up two sequences containing the range endpoints of $x(t)$ and $h(t)$.
- Form their pairwise sum (by summing each value from one sequence with the other).
- Arrange in increasing order & discard duplications.

Example 2.3

Two sequences for signals $h(t)$, and $x(t)$ are $\{0, 2\}$, and $\{-1, 0, 1\}$. Find the ranges for nonzero convolution. The pairwise sum is $\{-1, 0, 1, 1, 2, 3\}$. Discarding duplicates, we get $\{-1, 0, 1, 2, 3\}$. Therefore, the ranges for nonzero convolution are

$$-1 \leq t < 0, \quad 0 \leq t < 1, \quad 1 \leq t < 2, \quad \& \quad 2 \leq t < 3$$

Analytical Convolution

Given the analytical expressions for $x(t)$, and $h(t)$, for $x(t) * h(t)$, we form $x(\lambda)$, and $h(t - \lambda)$, or for $h(t) * x(t)$, we form $h(\lambda)$, and $x(t - \lambda)$, and integrate their products. Remember that when integrating $h(t - \lambda)$, and $x(t - \lambda)$ are function of λ , and t is constant with respect to λ . For piecewise signals the integral is considered over the limits of each interval.

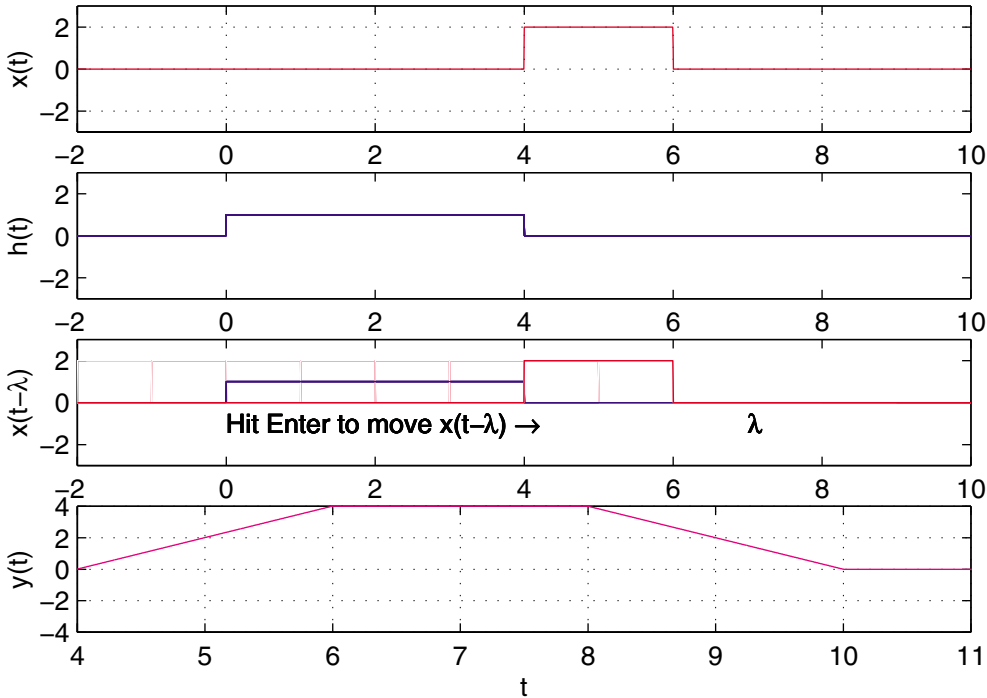
Example 2.4 (Example 2-7 Textbook)

Obtain the convolution of the two following rectangular-pulse signals.

$$x(t) = 2\Pi\left(\frac{t-5}{2}\right) \quad \text{and} \quad h(t) = \Pi\left(\frac{t-2}{4}\right)$$

Run `chs2_ex4.m` to see a demonstration of

$$y(t) = x(t) * h(t)$$



In the convolution integral, the time t , determines the relative location of $x(t - \lambda)$ with respect to $h(\lambda)$. We must evaluate the product $x(t - \lambda)h(\lambda)$ as t varies. First $x(t)$ is reversed and shifted to obtain $x(t - \lambda)$. Then the signal $x(t - \lambda)$ is moved along the t -axis, and the overlap area is evaluated. The pairwise sum of sequences $\{0, 4\}$, and $\{4, 6\}$ is $\{4, 6, 8, 10\}$. The ranges for nonzero convolution are

$$4 < t \leq 6, \quad 6 < t \leq 8, \text{ and } 8 < t \leq 10$$

. Thus, $y(t)$ is given by

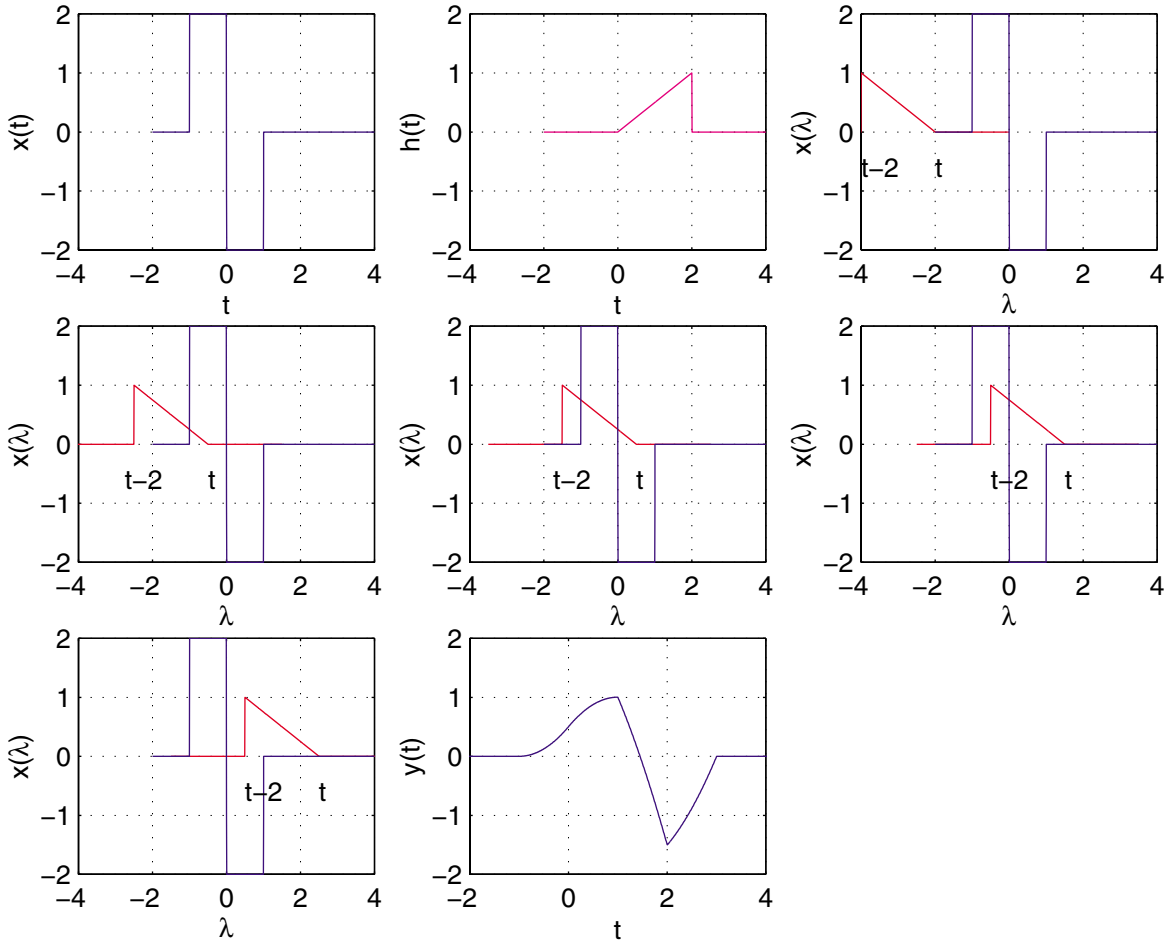
$$y(t) = \begin{cases} 2(t - 4) & 4 < t \leq 6 \\ 4 & 6 < t \leq 8 \\ 2[4 - (t - 6)] = 2(10 - t) & 8 < t \leq 10 \\ 0 & \text{otherwise} \end{cases}$$

Example 2.5 (Example 2-8 Textbook)
 Obtain the convolution of the two following signals.

$$x(t) = 2\Pi(t + 0.5) - 2\Pi(t - 0.5) \quad \text{and}$$

$$h(t) = 0.5r(t)u(2 - t)$$

Run **chs2_ex5** to see a demonstration of $y(t) = x(t) * h(t)$



The pairwise sum of the sequences $\{0, 2\}$, and $\{-1, 0, 1\}$ is $\{-1, 0, 1, 1, 2, 3\}$. Discarding duplicates, we get $\{-1, 0, 1, 2, 3\}$. Therefore, the ranges for nonzero convolution are

$$-1 \leq t < 0, \quad 0 \leq t < 1, \quad 1 \leq t < 2, \quad \& \quad 2 \leq t < 3$$

Referring to the illustrated graph, the integral is formed and evaluated at various intervals as follows

Since $h(t) = \frac{1}{2}t$, then $h(t - \lambda) = \frac{1}{2}(t - \lambda)$.

For $t < 0$, or $t > 3$ $y(t) = 0$

For $-1 \leq t < 0$

$$y(t) = \int_{-1}^t (2) \frac{1}{2} (t - \lambda) d\lambda = \int_{-1}^t (t - \lambda) d\lambda$$

To integrate, we make a change of variable, $\tau = t - \lambda$, therefore, $d\tau = -d\lambda$, and changing the limits of the integral, the result is

$$y(t) = - \int_{t+1}^0 \tau d\tau = - \frac{1}{2} \tau^2 \Big|_{t+1}^0 = \frac{1}{2} (t + 1)^2$$

For $0 \leq t < 1$

$$y(t) = \int_{-1}^0 (2) \frac{1}{2} (t - \lambda) d\lambda + \int_0^t (-2) \frac{1}{2} (t - \lambda) d\lambda$$

Change the variable, $\tau = t - \lambda$, therefore, $d\tau = -d\lambda$, and changing the limits of the integral, the result is

$$\begin{aligned} y(t) &= - \int_{t+1}^t \tau d\tau + \int_t^0 \tau d\tau \\ &= - \frac{1}{2} \tau^2 \Big|_{t+1}^t + \frac{1}{2} \tau^2 \Big|_t^0 \\ &= \frac{1}{2} (-t^2 + 2t + 1) \end{aligned}$$

For $1 \leq t < 2$

$$y(t) = \int_{t-2}^0 (2) \frac{1}{2} (t - \lambda) d\lambda + \int_0^1 (-2) \frac{1}{2} (t - \lambda) d\lambda$$

Change the variable, $\tau = t - \lambda$, therefore, $d\tau = -d\lambda$, and changing the limits of the integral, the result is

$$\begin{aligned} y(t) &= - \int_2^t \tau d\tau + \int_t^{t-1} \tau d\tau \\ &= - \frac{1}{2} \tau^2 \Big|_2^t + \frac{1}{2} \tau^2 \Big|_t^{t-1} \\ &= \frac{1}{2} (-t^2 - 2t + 1) + 2 \end{aligned}$$

For $2 \leq t < 3$

$$y(t) = \int_{t-2}^1 (-2) \frac{1}{2} (t - \lambda) d\lambda$$

Change the variable, $\tau = t - \lambda$, therefore, $d\tau = -d\lambda$, and changing the limits of the integral, the result is

$$\begin{aligned} y(t) &= \int_2^{t-1} \tau d\tau = \frac{1}{2} \tau^2 \Big|_2^{t-1} \\ &= \frac{1}{2} (t^2 - 2t + 1) - 2 \end{aligned}$$

Example 2.6

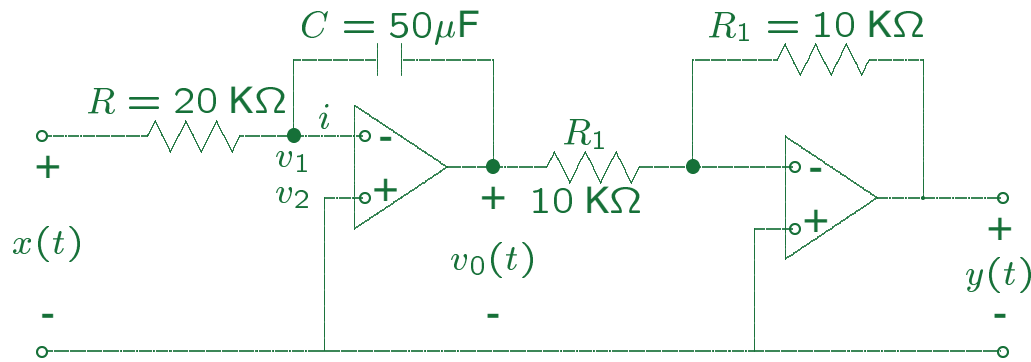
Given $x(t) = e^{-3t}u(t)$ and $h(t) = e^{-t}u(t)$ evaluate the convolution $y(t) = x(t) * h(t)$.

$x(\lambda) = e^{-3\lambda}u(\lambda)$ and $h(t - \lambda) = e^{-(t-\lambda)}u(t - \lambda)$. Since $u(\lambda) = 0$, for $\lambda < 0$, and $u(t - \lambda) = 0$, for $\lambda > t$, the convolution integral becomes

$$\begin{aligned}y(t) &= x(t) * h(t) = \int_{-\infty}^{\infty} e^{-3\lambda}u(\lambda)e^{-(t-\lambda)}u(t - \lambda)d\lambda \\ &= e^{-t} \int_0^t e^{-2\lambda}d\lambda = e^{-t} \left[-\frac{1}{2}e^{-2\lambda} \right]_0^t \\ &= e^{-t} \left(1 - \frac{1}{2}e^{-2t} \right)\end{aligned}$$

Example 2.7

In the ideal op amp circuit shown, show that the output voltage $y(t)$ is the integral of the input voltage $x(t)$.



Applying KCL to node 1, we have

$$\frac{v_1 - x(t)}{R} + C \frac{d}{dt}(v_1 - v_0(t)) = 0$$

For ideal op amps, $v_1 = v_2 = 0$, $i = 0$, and we obtain

$$-\frac{x(t)}{R} - C \frac{dv_0(t)}{dt} = 0 \quad \text{or} \quad v_0(t) = -\frac{1}{RC} \int_{-\infty}^t x(\lambda) d\lambda$$

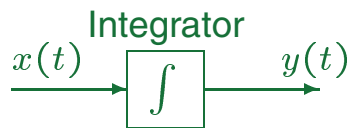
Substituting for R and C , we get $RC = 1$, and since

$$y(t) = -\frac{R_1}{R_1} v_0(t) \quad \text{we obtain}$$

$$y(t) = \int_{-\infty}^t x(\lambda) d\lambda$$

$$y(t) = \int_{-\infty}^t x(\lambda) d\lambda$$

The above integral is presented symbolically as



Example 2.8

(a) Obtain the impulse response of the integrator in Example 2.7. The input is a unit impulse, i.e., $x(t) = \delta(t)$, then the impulse response is

$$h(t) = \int_{-\infty}^t \delta(\lambda) d\lambda = u(t)$$

(b) Using the convolution integral find the system response to a unit ramp input, $x(t) = tu(t)$.

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} \lambda u(\lambda) u(t - \lambda) d\lambda$$

Since $u(\lambda) = 0$ for $\lambda < 0$, and $u(t - \lambda) = 0$, for $\lambda > t$, we have

$$y(t) = \int_0^t \lambda d\lambda = \frac{1}{2}t^2$$

More on the Impulse response of LTI Systems

Finding the impulse response of a system directly from time domain is often tedious compared to the solution in s-domain.

In s -domain, the response of a system described by a transfer function $G(s)$ is

$$Y(s) = G(s)X(s)$$

If the input is a unit impulse function, i.e., $x(t) = \delta(t)$, then $X(s) = \mathcal{L}[\delta(t)] = 1$, the s-domain impulse response is

$$H(s) = G(s)(1) \quad \text{or} \quad h(t) = \mathcal{L}^{-1}[G(s)]$$

Therefore, the inverse Laplace transform of the transfer function is the impulse response. In this section we consider finding the impulse response in time-domain.

If the step response of a system is known, then since $\delta(t) = \frac{du(t)}{dt}$, then for LTI systems the impulse response $h(t)$ is the derivative of the step response $u(t)$, i.e.,

$$h(t) = \frac{da(t)}{dt}$$

where $a(t)$ is the step response.

We have already obtained the impulse response for a first-order system (RC circuit). To generalize the technique, consider a first-order differential equation described by

$$\frac{dy(t)}{dt} + a_0y(t) = b_0x(t)$$

For impulse response, we write

$$\frac{dh(t)}{dt} + a_0h(t) = b_0\delta(t)$$

Assuming a solution of the form $h(t) = Ae^{pt}$ and substituting in the given equation, since for $t > 0$, $\delta(t) = 0$, we obtain

$$(p + a_0)Ae^{pt} = 0 \quad \Rightarrow \quad p = -a_0$$

Therefore,

$$h(t) = Ae^{-a_0t} \quad t > 0$$

To find $A = h(0^+)$, we need $h(0^+)$, so rewriting the given equation for $t \geq 0$

$$\frac{dh(t)}{dt} + a_0h(t) = b_0\delta(t) \quad t \geq 0$$

and the integral from $t = 0^-$ to $t = 0^+$ is

$$\int_{t=0^-}^{0^+} \frac{dh(t)}{dt} dt + \int_{t=0^-}^{0^+} a_0h(t) dt = b_0 \int_{t=0^-}^{0^+} \delta(t) dt$$
$$[h(0^+) - h(0^-)] + 0 = b_0$$

Since $h(0^-) = 0$, we have $h(0^+) = b_0$, therefore $A = b_0$, and the impulse response is

$$h(t) = b_0e^{-a_0t}$$

To generalize the solution, we conclude that the impulse response of

$$\frac{dy(t)}{dt} + a_0y(t) = b_0x(t) \quad \text{with } y(t) = 0$$

is equivalent to solving the homogeneous equation

$$\frac{dy(t)}{dt} + a_0y(t) = 0 \quad \text{with } y(0) = b_0$$

Similarly, the impulse response of a second order system

$$\frac{d^2y(t)}{dt^2} + a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_0 x(t)$$

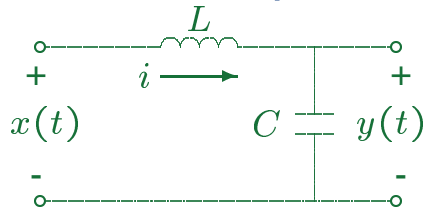
can be found as the solution to the homogeneous equation

$$\frac{d^2y(t)}{dt^2} + a_1 \frac{dy(t)}{dt} + a_0 y(t) = 0 \quad \text{with}$$

$$y(0) = 0 \quad \text{and} \quad \frac{dy(0)}{dt} = b_0$$

Example 2.9 (Example 2-12 textbook)

Find the impulse response of the LC circuit shown



Applying KVL, we have

$$L \frac{di}{dt} + y(t) = x(t) \quad \text{and}$$

$$i = C \frac{dy(t)}{dt} \quad \text{therefore, we have}$$

$$\frac{d^2 y(t)}{dt^2} + \frac{1}{LC} y(t) = \frac{1}{LC} x(t)$$

The impulse response is equivalent to the homogeneous solution given by

$$\frac{d^2 y(t)}{dt^2} + \frac{1}{LC} y(t) = 0 \quad \text{with } y(0) = 0, \& \quad y' = \frac{1}{LC}$$

The characteristic equation is

$$p^2 + \frac{1}{LC} = 0 \quad \text{or } p = \pm j \frac{1}{\sqrt{LC}} = \pm j\omega_0$$

Therefore, the impulse response is

$$h(t) = [B_1 \cos \omega_0 t + B_2 \sin \omega_0 t]u(t)$$

Applying the initial conditions

$$h(0) = 0 = B_1 \quad \text{and}$$

$$h'(0) = \omega_0 B_2 = \frac{1}{LC} \quad \text{or } B_2 = \omega_0$$

Therefore, the impulse response is

$$h(t) = \omega_0 \sin \omega_0 t u(t)$$

Superposition Integral in terms of Step Response

The convolution integral in terms of the impulse

$$y(t) = \int_{-\infty}^{\infty} x(t - \lambda)h(\lambda)d\lambda$$

Using integration by part $\int_a^b u dv = uv|_a^b - \int_a^b v du$. Let $u = x(t - \lambda)$ and $dv = h(\lambda)d\lambda$, then

$du = -\dot{x}(t - \lambda)d\lambda$ and

$v(\lambda) = \int_{-\infty}^{\lambda} h(\zeta)d\zeta = a(t)$, (step response)

Thus, we have

$$\begin{aligned} y(t) &= a(\lambda) x(t - \lambda)|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \dot{x}(t - \lambda)a(\lambda)d\lambda \\ &= 0 + \int_{-\infty}^{\infty} \dot{x}(t - \lambda)a(\lambda)d\lambda \end{aligned}$$

Thus, convolution of $x(t)$, with step response $a(t)$, known as *Duhamel's integral* is

$$y(t) = a(t) * x(t) = \int_{-\infty}^{\infty} a(\lambda)\dot{x}(t - \lambda)d\lambda$$

Alternatively,

$$y(t) = x(t) * a(t) = \int_{-\infty}^{\infty} \dot{x}(\lambda)a(t - \lambda)d\lambda$$

Example 2.10 (Example 12-13 Textbook)

The input to a system is a unit ramp, i.e., $x(t) = tu(t)$. Using the convolution integral in terms of step response find the ramp response.

Since $\dot{x}(t) = u(t)$, then

$$\begin{aligned}y_R(t) &= a(t) * x(t) = \int_{-\infty}^{\infty} a(\lambda)\dot{x}(t - \lambda)d\lambda \\ &= \int_{-\infty}^{\infty} a(\lambda)u(t - \lambda)d\lambda = \int_{-\infty}^t a(\lambda)d\lambda\end{aligned}$$

Thus, the ramp response is the integral of the step response.

Example 2.11 (Example 2-14 Text)

The step response of the RC circuit considered earlier is

$$a(t) = \left(1 - e^{\frac{-t}{\tau}}\right) u(t)$$

Find the response to the triangular signal

$$x_{\Delta} = r(t) - 2r(t - 1) + r(t - 2)$$

We first find the ramp response?

$$\begin{aligned} y_R(t) &= \int_0^t a(\lambda) d\lambda = \int_0^t \left(1 - e^{\frac{-\lambda}{\tau}}\right) u(\lambda) d\lambda \\ &= r(t) + \tau \left[e^{\frac{-\lambda}{\tau}} \right]_0^t u(t) \\ &= r(t) - \tau \left[1 - e^{\frac{-t}{\tau}} u(t) \right] \end{aligned}$$

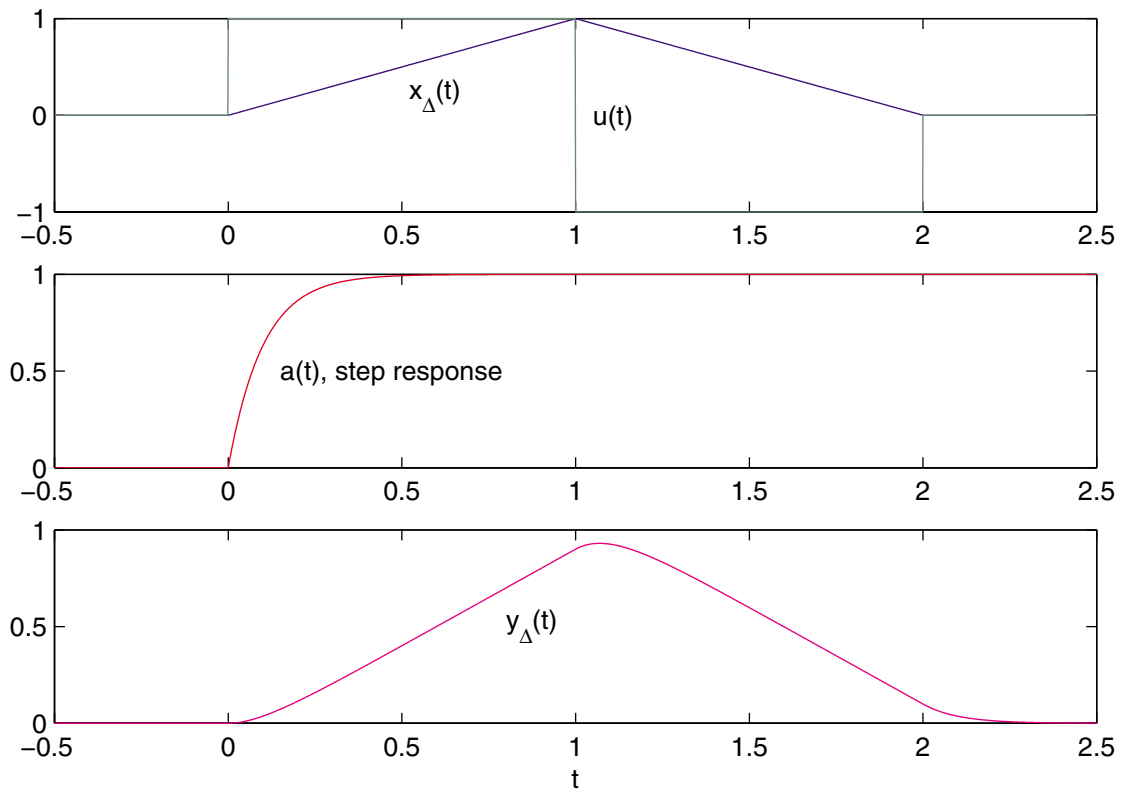
By superposition the response to $x_{\Delta}(t)$ signal is

$$y_{\Delta}(t) = y_R(t) - 2y_R(t - 1) + y_R(t - 2)$$

The following script file (**chs2_ex11.m**) demonstrate the Duhamel's integral for the above example (Example 2-14 Text)

```
% DuHammel's integral for Example 2-14 Text
%
RC = 0.1;
del_t = 0.001;
t = -0.5:del_t:2.5;
tp = -2*0.5:del_t:2*2.5;           % time range for output
x_del=rmp_fn(t)-2*rmp_fn(t-1)+rmp_fn(t-2);%triangular input
x_del_dot=stp_fn(t)-2*stp_fn(t-1)+stp_fn(t-2);%Der.of input
a = (1 - exp(-t/RC)).*stp_fn(t);   % step response
y = del_t*conv(x_del_dot, a);
subplot(3,1,1), plot(t, x_del, t, x_del_dot)
text(0.5, 0.2, ' x_{\Delta}(t)'), text(1.05, 0, 'u(t)')
subplot(3,1,2), plot(t, a, 'r'),
text(.15,.5, 'a(t), step response')
subplot(3,1,3), plot(tp, y, 'm'), xlabel('t'),
axis([-0.5 2.5 0 1])
text(0.8, .5, 'y_{\Delta}(t)')
```

The result is



Frequency Response Function of a LTI system

The sinusoidal input and its response is of major importance in signal analysis, since a periodic signal can be expressed as a weighted combination of sinusoidal signals. If we know the response of a LTI system to sinusoids of all frequencies, we have a complete characteristics of the system. In the Electric Circuits course, we learned that, if the input to a LTI system is a sinusoid of frequency ω rad/s, the steady-state response (i.e., response after all transient has decayed) is a sinusoid of the same frequency.

Consider a complex sinusoidal input $x(t) = e^{j\omega t}$ to a system with its impulse response given by $h(t)$. The convolution integral gives the output as

$$y(t) = \int_{-\infty}^{\infty} e^{j\omega(t-\lambda)} h(\lambda) d\lambda$$

Since the integral is with respect to λ , we can write

$$\begin{aligned} y(t) &= e^{j\omega t} \int_{-\infty}^{\infty} h(\lambda) e^{-j\omega\lambda} d\lambda \\ &= e^{j\omega t} H(\omega) \end{aligned}$$

where, we have defined

$$H(\omega) = \int_{-\infty}^{\infty} h(\lambda) e^{-j\omega\lambda} d\lambda$$

The system output is a complex sinusoid of the same frequency as the input multiplied by the complex function $H(\omega)$. We shall learn that $H(\omega)$ corresponds to the Fourier transform of the impulse response $h(t)$.

Expressing the response in polar form, we have

$$y(t) = |H(\omega)|e^{j(\omega t + \angle H(\omega))}$$

where $|H(\omega)|$ is termed the *magnitude response*, and $\angle H(\omega)$ is termed the *phase response* of the system.

For a sinusoidal input

$$x(t) = X \cos(\omega t + \phi)$$

The steady-state sinusoidal response is

$$y(t) = |H(\omega)|X \cos(\omega t + \phi + \angle H(\omega))$$

Example 2.12 (Example 2-16 Text)

Determine the frequency response function $H(\omega)$ for the RC circuit considered earlier.

The impulse response of the RC circuit was found to be

$$h(t) = \frac{1}{\tau} e^{-\frac{t}{\tau}} u(t)$$

Therefore, the FRTF is

$$\begin{aligned} H(\omega) &= \int_{-\infty}^{\infty} h(\lambda) e^{-j\omega\lambda} d\lambda = \int_{-\infty}^{\infty} \frac{1}{\tau} e^{-\frac{t}{\tau}} e^{-j\omega t} u(t) dt \\ &= \int_0^{\infty} \frac{1}{\tau} e^{-(j\omega + \frac{1}{\tau})t} dt = \left[-\frac{\frac{1}{\tau}}{j\omega + \frac{1}{\tau}} e^{-(j\omega + \frac{1}{\tau})t} \right]_0^{\infty} \\ &= \frac{1}{1 + j\omega\tau} = \frac{1}{\sqrt{1 + (\omega\tau)^2}} e^{-j \tan^{-1} \omega\tau} \end{aligned}$$

For the complex input signal

$$x(t) = X e^{j\omega t}$$

the output is

$$y(t) = \frac{X}{\sqrt{1 + (\omega\tau)^2}} e^{j(\omega t - \tan^{-1} \omega\tau)}$$

For the cosine input $x(t) = X \cos(\omega t)$, the response is

$$y(t) = \frac{X}{\sqrt{1 + (\omega\tau)^2}} \cos(\omega t - \tan^{-1} \omega\tau)$$

System Stability

A system is *bounded-input, bounded-output stable*, if for every bounded input, the output remains bounded for all times. For a LTI system described by the differential equation

$$\frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \dots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_m \frac{d^m x(t)}{dt^m} + b_{m-1} \frac{d^{m-1} x(t)}{dt^{m-1}} + \dots + b_1 \frac{dx(t)}{dt} + b_0 x(t)$$

is BIBO stable if the $m < n$, and all roots of the characteristic polynomial

$$p^n + a_{n-1} p^{n-1} + \dots + a_1 p + a_0 = 0$$

have negative real part.

By definition an input signal $x(t)$ is bounded if there exist a number M such that

$$|x(t)| \leq M \quad \text{for all } t$$

Then a system is BIBO stable if, for a finite number R ,

$$|y(t)| \leq R \quad \text{for all } t$$

For example, the response given by

$$y(t) = x^2(t)$$

is BIBO stable, because $y(t)$ is bounded for any bounded input $x(t)$. On the other hand the integrating op amp

$$y(t) = \int_0^t x(\lambda) d\lambda$$

is unstable. Let $h(t) = u(t)$

$$\int_0^{\infty} |h(t)| dt = \int_0^{\infty} 1 dt = t \Big|_0^{\infty} = \infty \quad \Rightarrow \quad \text{unbounded}$$

Since the input-output characteristics of a LTI system are completely described by its impulse response $h(t)$, then expressing the response $y(t)$ in terms of the convolution integral

$$y(t) = \int_{-\infty}^{\infty} x(\lambda)h(t - \lambda)d\lambda$$

for bounded input

$$|x(t)| \leq M < \infty$$

the output

$$\begin{aligned} |y(t)| &= \left| \int_{-\infty}^{\infty} x(\lambda)h(t - \lambda)d\lambda \right| \leq \int_{-\infty}^{\infty} M|h(t - \lambda)|d\lambda \\ &= M \int_{-\infty}^{\infty} |h(t - \lambda)|d\lambda = M \int_{-\infty}^{\infty} |h(\sigma)|d\sigma \end{aligned}$$

Therefore, the output is bounded if

$$\int_{-\infty}^{\infty} |h(t)|dt < \infty$$

Thus a system is BIBO if its impulse response is absolutely integrable.

Example 2.13 (Example 2-17 Text)

Is the system described by the RC circuit stable?

The impulse response of the RC circuit was found as

$$h(t) = \frac{1}{\tau} e^{-\frac{t}{\tau}} u(t)$$

For BIBO

$$\begin{aligned} \int_{-\infty}^{\infty} \left| \frac{1}{\tau} e^{-\frac{t}{\tau}} u(t) \right| dt &= \int_0^{\infty} \frac{1}{\tau} e^{-\frac{t}{\tau}} dt \\ &= - \left[e^{-\frac{t}{\tau}} \right]_0^{\infty} = 1 < \infty \end{aligned}$$

There the RC circuit is BIBO stable.

Example 2.14 (Example 2-18 Textbook)

The step response of Example 2.9 (Example 2-12 textbook) was found to be

$$h(t) = \sin \omega_0 t u(t)$$

Determine whether the system is stable or unstable.

$$\int_{-\infty}^{\infty} |h(t)| dt = \omega_0 \int_{-\infty}^{\infty} |\sin \omega_0 t u(t)| dt = \omega_0 \int_0^{\infty} |\sin \omega_0 t| dt$$

The integral does not converge, and the system is BIBO unstable.

Example 2.15

The step response of a system is

$$h(t) = e^{4t}u(t)$$

Determine whether the system is stable or unstable.

$$\int_{-\infty}^{\infty} |h(t)| dt = \int_{-\infty}^{\infty} e^{4t}u(t) dt = \int_0^{\infty} e^{4t} dt$$

The integral is unbounded at the upper limit and the system is BIBO unstable.

SIMULATION DIAGRAM

The differential equations of a lumped linear network can be written in the form

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t)\end{aligned}\tag{1}$$

This system of first-order differential equations is known as the state equation of the system, and \mathbf{x} is the state vector. One advantage of the state-space method is that the form lends itself easily to the digital and/or analog computer methods of solution. Further, the state-space method can be easily extended to analysis of nonlinear systems. State equations may be obtained from an n th-order differential equation or directly from the system model by identifying appropriate state variables.

To illustrate how we select a set of state variables, consider an n th-order linear plant model described by the differential equation

$$\frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_1 \frac{dy}{dt} + a_0 y = u(t) \quad (2)$$

where $y(t)$ is the plant output and $u(t)$ is its input. A state model for this system is not unique, but depends on the choice of a set of state variables. A useful set of state variables, referred to as *phase variables*, is defined as

$$x_1 = y, \quad x_2 = \dot{y}, \quad x_3 = \ddot{y}, \quad \dots, \quad x_n = y^{n-1}$$

We express $\dot{x}_k = x_{k+1}$ for $k = 1, 2, \dots, n-1$, and then solve for $d^n y/dt^n$, and replace y and its derivatives by the corresponding state variables to give

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= -a_0 x_1 - a_1 x_2 - \dots - a_{n-1} x_n + u(t) \end{aligned} \quad (3)$$

or in matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t) \quad (4)$$

and the output equation is

$$y = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \end{bmatrix} \mathbf{x} \quad (5)$$

The M-file **ode2phv.m** is developed which converts an n th-order ordinary differential equation to the state-space phase variable form. **[A, B, C] = ode2phv(ai, k)** returns the matrices **A, B, C**, where **ai** is a row vector containing coefficients of the equation in descending order, and **k** is the coefficient of the right-hand side.

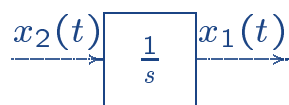
Equation (3) indicates that state variables are determined by integrating the corresponding state equation. A diagram known as the simulation diagram can be constructed to model the given differential equations. The basic element of the simulation diagram is the integrator. The first equation in (3) is

$$\dot{x}_1 = x_2$$

Integrating, we have

$$x_1 = \int x_2 dx$$

The above integral is shown by the following time-domain symbol. The integrating block is identified by symbol $\frac{1}{s}$. Adding an integrator for the remaining state variables and completing the last equation in (3) via a summing point and feedback paths, a simulation diagram is obtained.



INTRODUCTION TO SIMULINK

SIMULINK is an interactive environment for modeling, analyzing, and simulating a wide variety of dynamic systems. *SIMULINK* provides a graphical user interface for constructing block diagram models using “drag-and-drop” operations. A system is configured in terms of block diagram representation from a library of standard components. *SIMULINK* is very easy to learn. A system in block diagram representation is built easily and the simulation results are displayed quickly.

Simulation algorithms and parameters can be changed in the middle of a simulation with intuitive results, thus providing the user with a ready access learning tool for simulating many of the operational problems found in the real world. *SIMULINK* is particularly useful for studying the effects of nonlinearities on the behavior of the system, and as such, it is also an ideal research tool. The key features of *SIMULINK* are

- Interactive simulations with live display.
- A comprehensive block library for creating linear, nonlinear, discrete or hybrid multi-input/output systems.
- Seven integration methods for fixed-step, variable-step, and stiff systems.
- Unlimited hierarchical model structure.
- Scalar and vector connections.
- Mask facility for creating custom blocks and block libraries.

SIMULINK provides an open architecture that allows you to extend the simulation environment:

- You can easily perform “what if” analyses by changing model parameters – either interactively or in batch mode – while your simulations are running.
- Creating custom blocks and block libraries with your own icons and user interfaces from *MATLAB*, Fortran, or C code.
- You can generate C code from *SIMULINK* models for embedded applications and for rapid prototyping of control systems.
- You can create hierarchical models by grouping blocks into subsystems. There are no limits on the number of blocks or connections.
- *SIMULINK* provides immediate access to the mathematical, graphical, and programming capabilities of *MATLAB*, you can analyze data, automate procedures, and optimize parameters directly from *SIMULINK*.

SIMULATION PARAMETERS AND SOLVER

You set the simulation parameters and select the solver by choosing **Parameters** from the Simulation menu. *SIMULINK* displays the **Simulation Parameters** dialog box, which uses three “pages” to manage simulation parameters. **Solver**, **Workspace I/O**, and **Diagnostics**.

SOLVER PAGE The Solver page appears when you first choose **Parameters** from the **Simulation menu**. The Solver page allows you to:

- Set the start and stop times – The default start time is 0.0 seconds and the default stop time is 10.0 seconds.
- Choose the solver and specify solver parameters – You can choose between variable-step and fixed-step solvers. These are **ode45**, **ode23**, **ode113**, **ode15s**, **ode23s**, and **discrete**. The default is **ode45**. For variable-step solvers, you can set the maximum and initial step size. By default, these parameters are automatically determined. For fixed-step solvers, you can choose **ode5**, **ode4**, **ode3**, **ode2**, **ode1**, and **discrete**.
- Output Options – The Output options area of the dialog box enables you to control how much output the simulation generates. You can choose from three popup options. These are: Refine output, Produce additional output, and Produce specified output only.

WORKSPACE I/O PAGE

The Workspace I/O page manages the input from and the output to the *MATLAB* workspace, and allows:

- Loading input from the workspace – Input can be specified either as *MATLAB* command or as a matrix for the Import blocks.
- Saving the output to the workspace – You can specify return variables by selecting the Time, State, and/or Output check boxes in the Save to workspace area.

DIAGNOSTICS PAGE

The Diagnostics page allows you to select the level of warning messages displayed during a simulation.

THE SIMULATION PARAMETERS DIALOG BOX

Table below summarizes the actions performed by the dialog box buttons, which appear on the bottom of each dialog box page.

Button	Action
Apply	Applies the current parameter values and keeps the dialog box open. During a simulation, the parameter values are applied immediately.
Revert	Changes the parameter values back to the values they had when the Dialog box was most recently opened and applies the parameters.
Help	Displays help text for the dialog box page.
Close	Applies the parameter values and closes the dialog box. During a simulation, the parameter values are applied immediately.

To stop a simulation, choose Stop from the Simulation menu. The keyboard shortcut for stopping a simulation is Ctrl-T. You can suspend a running simulation by choosing Pause from the Simulation menu. When you select Pause, the menu item changes to Continue. You proceed with a suspended simulation by choosing Continue.

BLOCK DIAGRAM CONSTRUCTION

At the *MATLAB* prompt, type *SIMULINK*. The *SIMULINK* BLOCK LIBRARY, containing seven icons, and five pull-down menu heads, appears. Each icon contains various components in the titled category. To see the content of each category, double click on its icon. The easy-to-use pull-down menus allow you to create a *SIMULINK* block diagram, or open an existing file, perform the simulation, and make any modifications. Basically, one has to specify the model of the system (state space, discrete, transfer functions, nonlinear ode's, etc), the input (source) to the system, and where the output (sink) of the simulation of the system will go. Generally when building a model, design it first on the paper, then build it using the computer. When you start putting the blocks together into a model, add the blocks to the model window before adding the lines that connect them. This way, you can reduce how often you need to open block libraries. An introduction to *SIMULINK* is presented by constructing the *SIMULINK* diagram for the following examples.

EE-303 H. Saadat

MODELING EQUATIONS

Here are some examples that may improve your understanding of how to model equations.

Example A.21

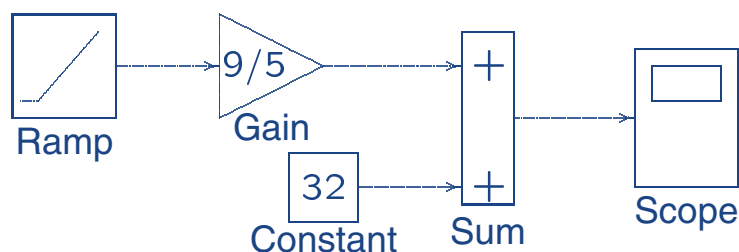
Model the equation that converts Celsius temperature to Fahrenheit. Obtain a display of Fahrenheit-Celsius temperature graph over a range of 0 to 100°C.

$$T_F = \frac{9}{5}T_C + 32 \quad (6)$$

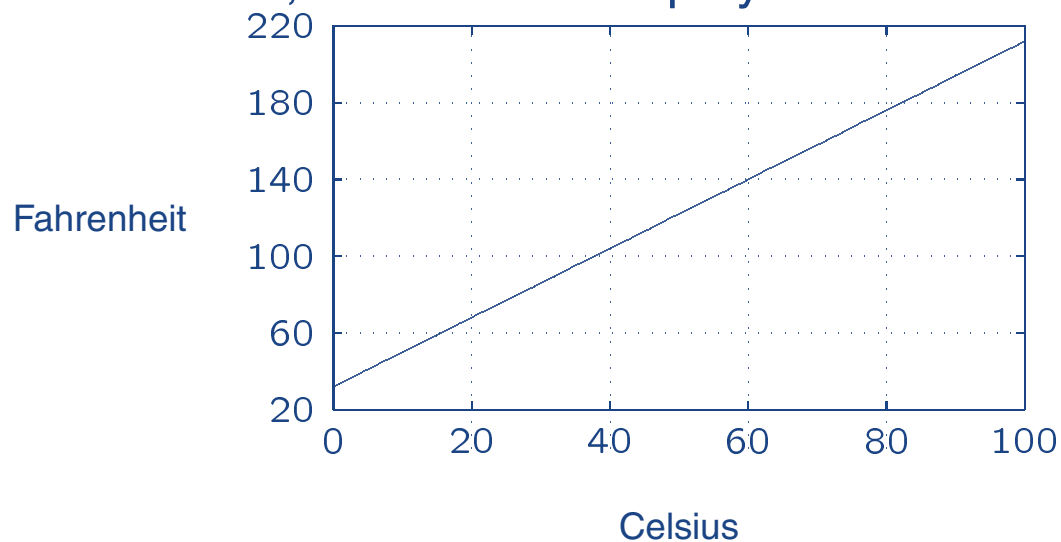
First, consider the blocks needed to build the model.

- A ramp block to input the temperature signal, from the source library.
- A constant block, to define the constant of 32, also from the source library.
- A gain block, to multiply the input signal by 9/5, from the Linear library.
- A sum block, to add the two quantities.
- A scope block to display the output, from the sink library.

To create a *SIMULINK* block diagram presentation select **new...** from the **File** menu. This provides an untitled blank window for designing and simulating a dynamic system. Copy the above blocks from the block libraries into the new window by depressing the mouse button and dragging. Assign the parameter values to the Gain and Constant blocks by opening (double clicking on) each block and entering the appropriate value. Then click on the **close** button to apply the value and close the dialog box. The next step is to connect these icons together by drawing lines connecting the icons using the left mouse button (hold the button down and drag the mouse to draw a line). You should have the block diagram shown.



The Ramp block inputs Celsius temperature. Open this block, set the Slope to 1, Start time to 0, and the Initial output to 0. The Gain block multiplies that temperature by the constant $9/5$. The sum block adds the value 32 to the result and outputs the Fahrenheit temperature. Pull down the Simulation dialog box and select Parameters. Set the Start time to zero and the Stop Time to 100. Pull down the **File** menu and use **Save** to save the model under **simexa21** Start the simulation. Double click on the Scope, click on the **Auto Scale**, the result is displayed as shown.



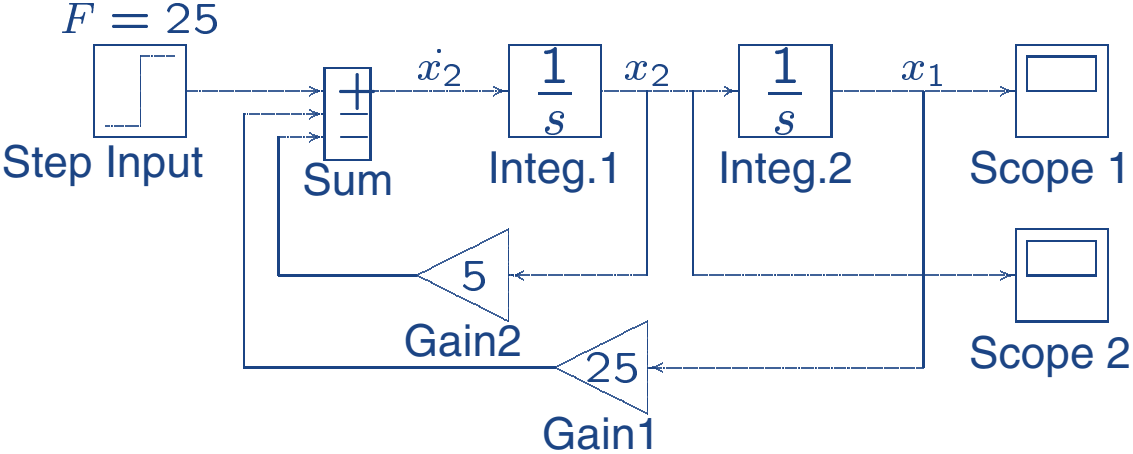
Example A.22

Construct a simulation diagram for the state equation described in Example A.18. Use *SIMULINK* to model and simulate the step response of this system.

State equation in Example A.18 for $M = 1$ kg, $B = 5$ N/m/sec, $K = 25$ N/m, and $f(t) = 25u(t)$, is

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -25x_1 - 5x_2 + 25u(t) \end{aligned}$$

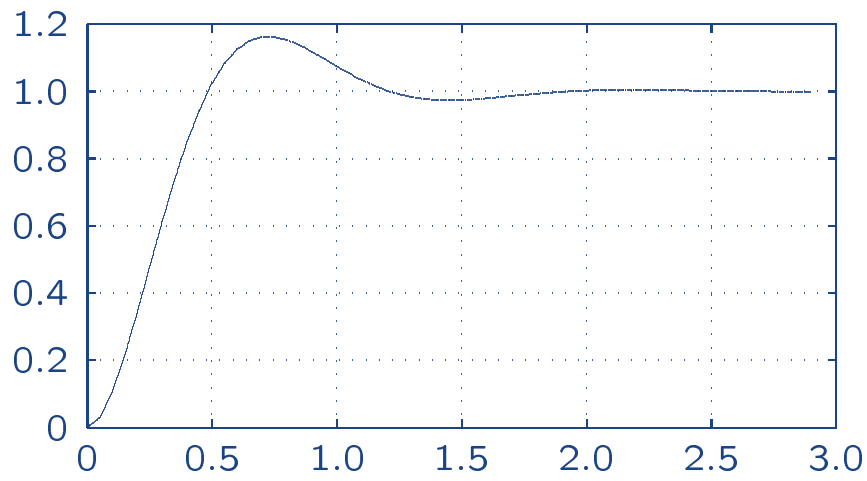
The simulation diagram is drawn from the above equations by inspection and is shown.



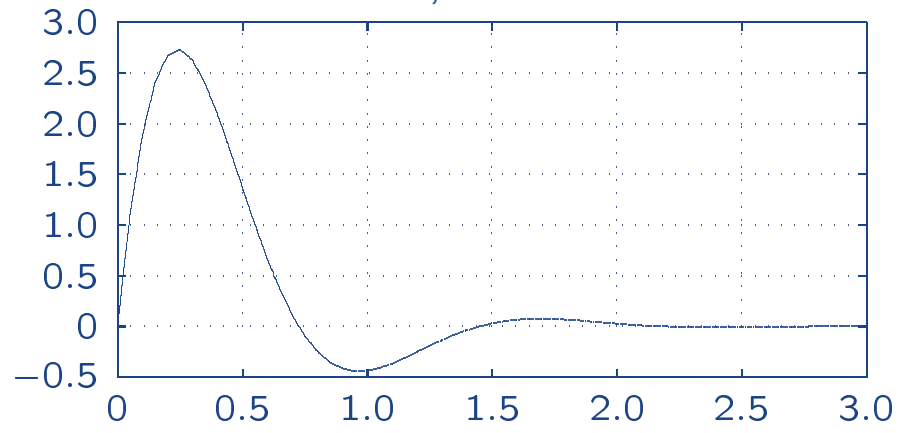
To create a *SIMULINK* block diagram presentation select **new...** from the **File** menu. This provides an untitled blank window for designing and simulating a dynamic system. You can copy blocks from within any of the seven block libraries or other previously opened windows into the new window by depressing the mouse button and dragging. Open the **Source Library** and drag the Step Input block to your window. Double click on Step Input to open its dialog box. Set the step time and the Initial Value to 0, and the Final Value to 25 to represent the step input. Open the **Linear Library** and drag the **Sum** block to the right of the Step Input block. Open the Sum dialog box and enter + - - under List of Signs. Using the left mouse button, click and drag from the Step output port to the Summing block input port to connect them. Drag a copy of the Integrator block from the **Linear Library** and connect it to the output port of the Sum block. Click on the Integrator block once to highlight it. Use the Edit command from the menu bar to copy and paste a second Integrator.

Next drag a copy of the Gain block from the **Linear Library**. Highlight the Gain block, and from the pull-down **Options** menu, click on the Flip Horizontal to rotate the Gain block by 180° . Double click on Gain block to open its dialog box and set the gain to 5. Make a copy of this block and set its gain to 25. Connect the output ports of the Gain blocks to the Sum block and their input ports to the locations shown. Finally, get two Auto-Scale Graphs from the **Sink Library**, and connect them to the output of each Integrator. Before starting simulation, you must set the simulation parameters. Pull down the **Simulation** dialog box and select **Parameters**. Set the Start Time to zero, the Stop Time to 3, and for a more accurate integration, set the Maximum Step Size to 0.1. Leave the other parameters at their default values. Press OK to close the dialog box.

If you don't like some aspect of the diagram, you can change it in a variety of ways. You can move any of the icons by clicking on its center and dragging. You can move any of the lines by clicking on one of its corners and dragging. You can change the size and the shape of any of the icons by clicking and dragging on its corners. You can remove any line or icon by clicking on it to select it and using the **cut** command from the **edit** menu. You should now have exactly the same system as shown. Pull down the **File** menu and use **Save as** to save the model under a file name **simexa22**. Start the simulation. *SIMULINK* will create the Figure windows and display the system responses. To see the second Figure window, click and drag the first one to a new location. The simulation results are shown in the next page.



t , sec



t , sec

Example A.23

Consider the system defined by

$$2\frac{d^3y}{dt^3} + 4\frac{d^2y}{dt^2} + 8\frac{dy}{dt} + 10y = 10u(t)$$

We have three state variables.

$$x_1 = y$$

$$x_2 = \dot{y}$$

$$x_3 = \ddot{y}$$

Then we obtain

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

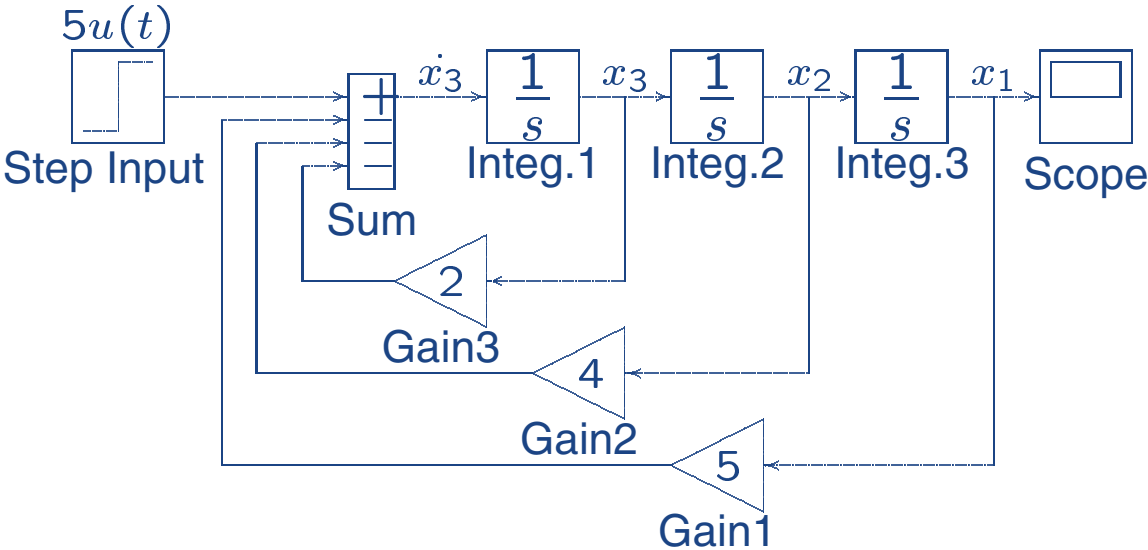
$$\dot{x}_3 = -5x_1 - 4x_2 - 2x_3 + 5u(t)$$

In matrix notation, the state and output equation are

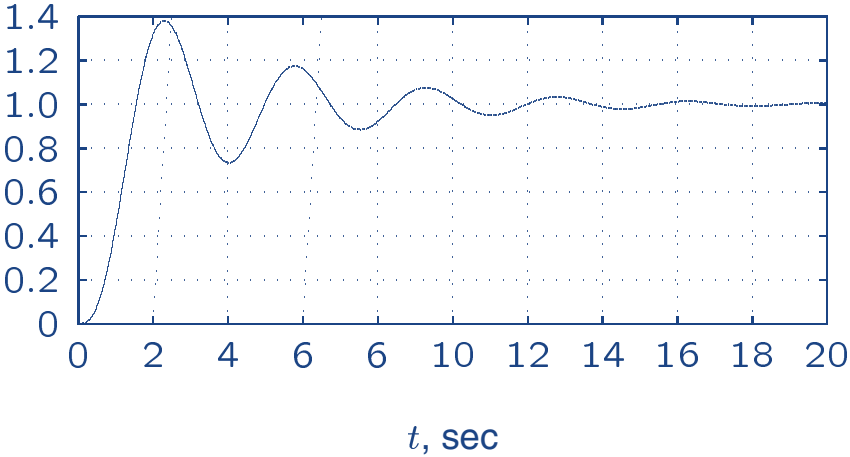
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x}$$

The simulation block diagram is obtained from the system differential equations as shown, it is saved as **simexa23**.



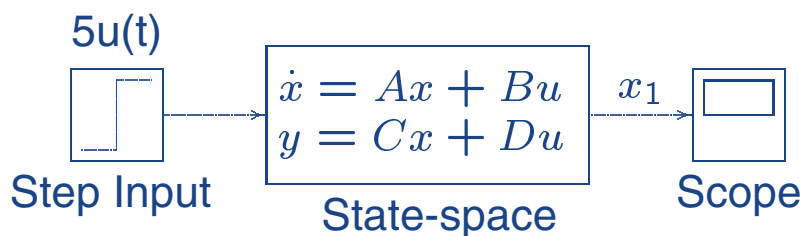
The simulation response is



Example A.24

Use the **state-space** model to simulate the state and output equations described in Example A.23.

The **State-Space** model provides a dialog box where the A , B , C , and D matrices can be entered in *MATLAB* matrix notation. A *SIMULINK* diagram using the **State-Space** model is constructed as shown, and is saved as **simexa24**. The result is the same as before.



Note that the output is given by $y = x_1$, and we define C as $C = [1 \quad 0 \quad 0]$. If it is desired to access all the states, then we can define C as an identity matrix, in this case a third order, i.e., $C = \text{eye}(3)$, and D as $D = \text{zeros}(3, 1)$. The output is a vector of state variables. A **DeMux** block may be added to produce individual states for graphing separately.

USING THE TO WORKSPACE BLOCK

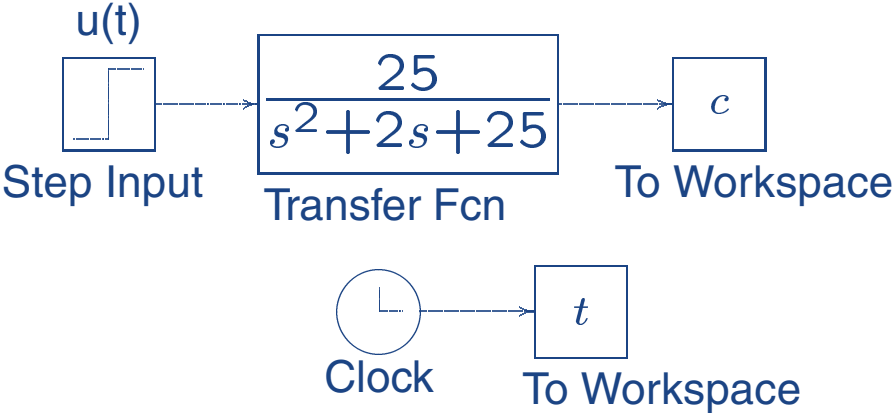
The To Workspace block can be used to return output trajectories to the *MATLAB* Workspace. This is illustrated below.

Example A.25

Obtain the step response of the following transfer function, and send the result to the *MATLAB* Workspace.

$$\frac{C(s)}{R(s)} = \frac{25}{s^2 + 2s + 25}$$

where $r(t)$ is a unit step function. The *SIMULINK* block diagram is constructed and saved in a file named **simexa25** as shown.



The To Workspace block can accept a vector input, with each input element's trajectories stored as a column vector in the resulting workspace variable. To specify the variables open the To Workspace block and for the variable name enter c . The time vector is stored by feeding a Clock block into To Workspace block. For this block variable name specify t . The vectors c and t are returned to *MATLAB* Workspace upon simulation.

LINEAR STATE-SPACE MODEL FROM SIMULINK DIAGRAM

SIMULINK provides the **linmod**, and **dlinmod** functions to extract linear models from the block diagram model in the form of the state-space matrices A, B, C, and D. State-space matrices describe the linear input-output relationship as

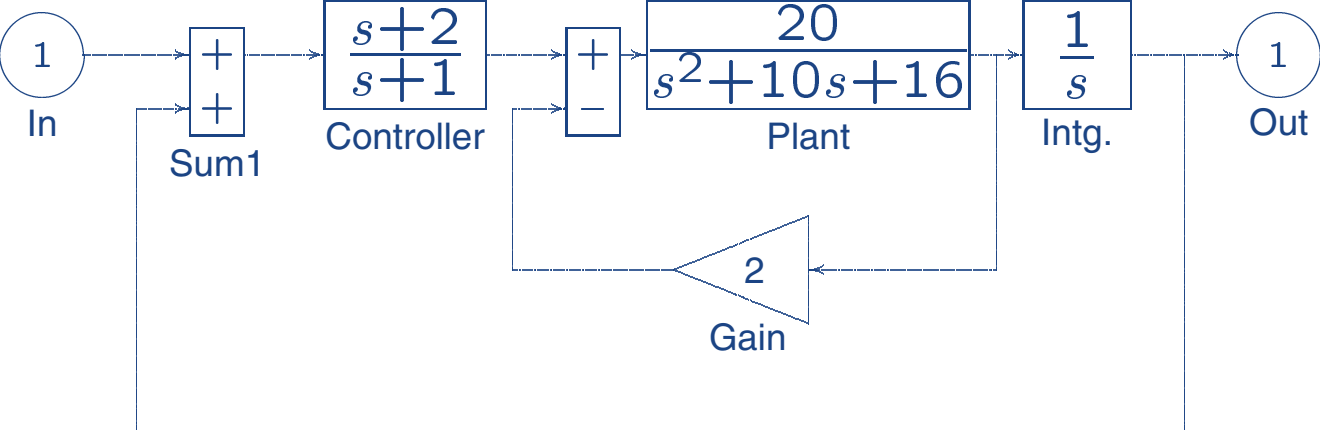
$$\dot{x}(t) = Ax(t) + Bu(t) \quad (7)$$

$$y(t) = Cx(t) + Du(t) \quad (8)$$

The following Example illustrates the use of **linmod** function. The input and outputs of the *SIMULINK* diagram must be defined using **Inport** and **Outport** blocks in place of the **Source** and **Sink** blocks.

Example A.26

Obtain the state-space model for the system represented by the block diagram shown below. The model is saved with a filename **simexa26**.



Run the simulation and to extract the linear model of this *SIMULINK* system, in the Command Window, enter the command

```
[A,B,C,D] = linmod('simexa26')
```

The result is

```
A =          0          0          0         20
      -1        -1          0          0
      -1         1        -10        -56
          0          0          1          0
C =
      1          0          0          0
D =
      0
```

In order to obtain the transfer function of the system from the state-space model, we use the command

```
[num, den]=ss2tf(A, B, C, D)
```

the result is

```
num =
      0.0000      0.0000      0.0000     20.0000     40.0000
den =
      1.0000     11.0000     66.0000     76.0000     40.0000
```

Thus, the transfer function model is

$$T(s) = \frac{20s + 40}{s^4 + 11s^3 + 66s^2 + 76s + 40}$$

Once the data is in the state-space form, or converted to a transfer function model, you can apply functions in Control System Toolbox for further analysis:

- Bode phase and magnitude frequency plot:

`bode(A, B, C, D)` or `bode(num, den)`

- Linearized time response:

`step(A, B, C, D)` or `step(num, den)`

`lsim(A, B, C, D)` or `lsim(num, den)`

`impulse(A, B, C, D)` or `impulse(num, den)`

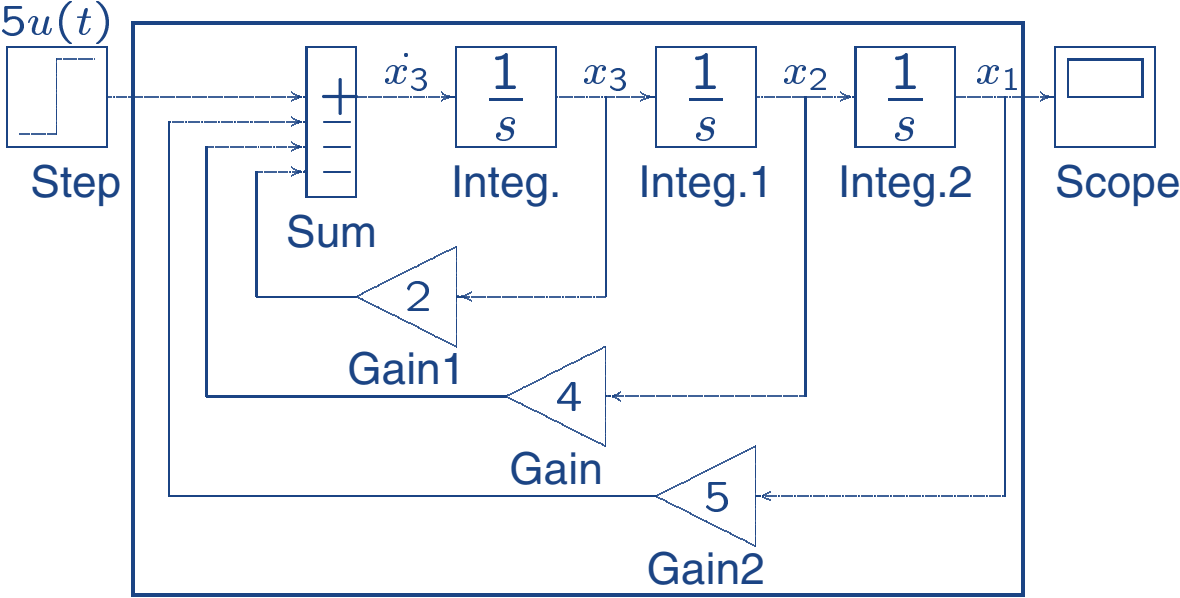
SUBSYSTEMS AND MASKING

SIMULINK subsystems, provide a capability within *SIMULINK* similar to subprograms in traditional programming languages.

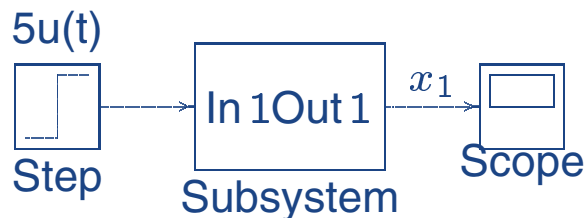
Masking is a powerful *SIMULINK* feature that enables you to customize the dialog box and icon for a block or subsystem. With masking, you can simplify the use of your model by replacing many dialog boxes in a subsystem with a single one.

Example A.27

To encapsulate a portion of an existing *SIMULINK* model into a subsystem, consider the *SIMULINK* model of Example A.23 as shown, and proceed as follows:



1. Select all the blocks and signal lines to be included in the subsystem with the bounding box as shown.
2. Choose Edit and select Create Subsystem from the model window menu bar. *SIMULINK* will replace the select blocks with a subsystem block that has an input port for each signal entering the new subsystem and an output port for each signal leaving the new subsystem. *SIMULINK* will assign default names to the input and output ports.



To mask a block, select the block, then choose Create Mask from the Edit menu. The Mask Editor appears. The Mask Editor consists of three pages, each handling a different aspect of the mask.

- The Initialization page enables you to define and describe mask dialog box parameter prompts, name the variables associated with the parameters, and specify initialization commands.
- The Icon page enables you to define the block icon.
- The Documentation page enables you to define the mask type, and specify the block description and the block help.

In this example for icon the system transfer function is entered with command

```
dpoly([10], [2 4 8 10])
```

A short description of the system and relevant help topics can be entered in the Documentation page. The subsystem block is saved in a file named **simexa28**. Many interesting examples are available in *SIMULINK demo*.

Fourier Series

In the previous chapter, we expressed an input signal as a weighted superposition of time-shifted impulses, and we found that the output is given by a weighted superposition of the impulse responses. The expression for the output was termed *Convolution*. The solution required the knowledge of the impulse response and determination of the integral. A task that is not always simple.

In this chapter, we consider approximating a periodic signal as a weighted superposition of harmonically related sinusoidal signals. This is one of the interesting parts of the famous Joseph Fourier thesis in 1807.

Fourier transform and analysis is widely used in signal analysis and other fields. Fourier transform changes a time-domain signal $x(t)$ into the frequency domain $X(f)$, allowing the analysis in the frequency domain with ease. To be studied in Chapter 4.

Before presenting the general Fourier series, let us see if we can represent a rectangular or a triangular signal as a series sum of harmonically related sine or cosine signals. To demonstrate this we consider two examples from the textbook.

Example 3.1 (Example 3-1 Textbook) Consider the partial sum of the following periodic signals.

$$\begin{aligned}x(t) &= \sin \omega_0 t + \frac{1}{3} \sin 3\omega_0 t + \frac{1}{5} \sin 5\omega_0 t + \dots \\ &= \sum_{n=1}^{\infty} \frac{1}{n} \sin n\omega_0 t \quad n \text{ is an odd integer}\end{aligned}$$

where, $T_0 = \frac{2\pi}{\omega_0}$ is the fundamental period.

The convergence of the series can be checked at any point, e.g., at $\omega_0 t = \frac{\pi}{2}$, we have

$$x\left(\frac{\pi}{2}\right) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$$

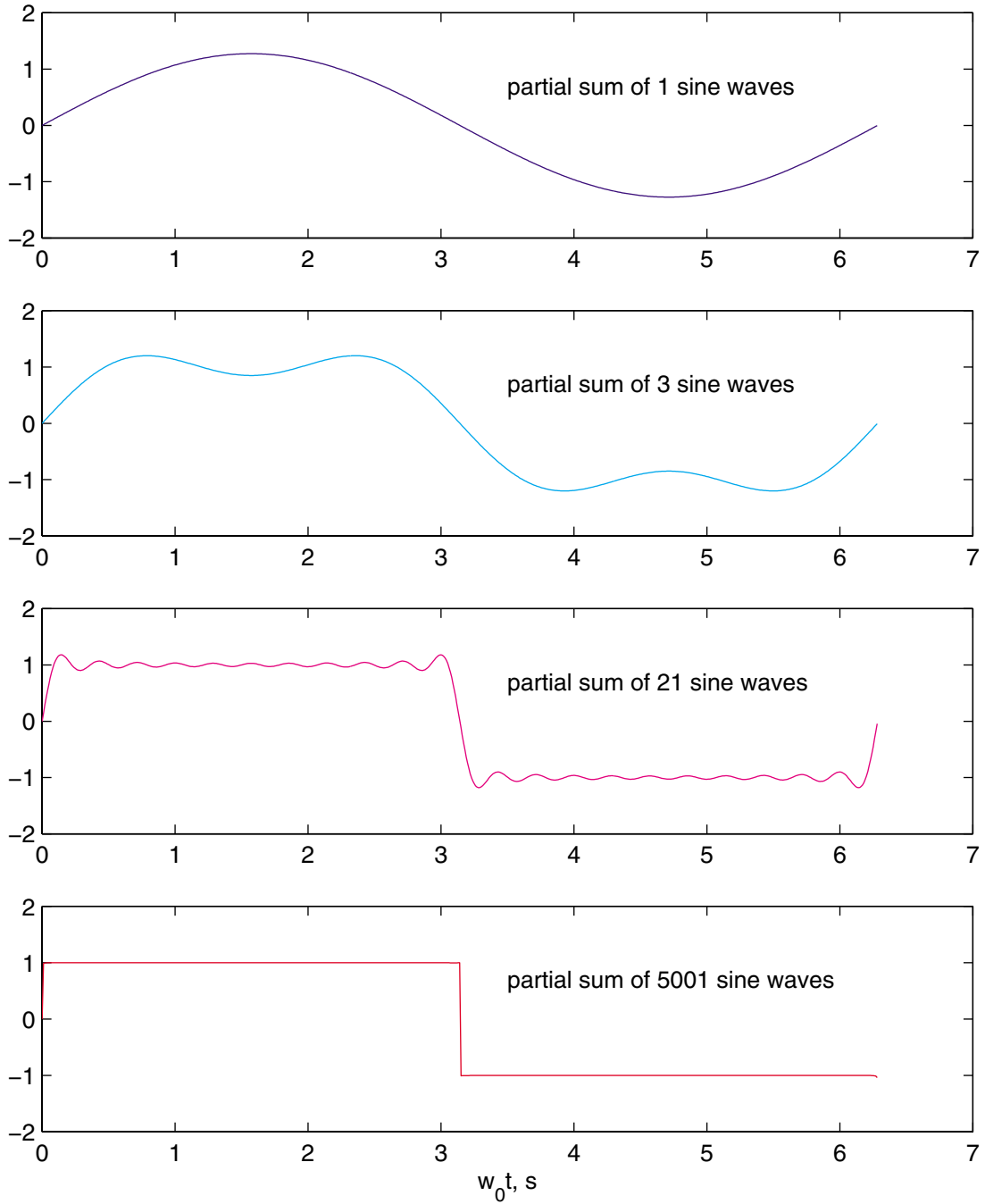
The convergence of the series to $\frac{\pi}{4}$ can be checked from the table of sum of series.

We can normalize the above series to have a value of unity at $\omega_0 t = \frac{\pi}{2}$ by multiplying it by $\frac{4}{\pi}$, i.e.,

$$\begin{aligned} x(t) &= \frac{4}{\pi} \left[\sin \omega_0 t + \frac{1}{3} \sin 3\omega_0 t + \frac{1}{5} \sin 5\omega_0 t + \dots \right] \\ &= \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin n\omega_0 t \quad n \text{ is an odd integer} \end{aligned}$$

Let us write a few simple MATLAB statements to generate this sum and its plot for any given odd integer.

```
%chs3_ex1.m
n = input('Enter an odd integer ');
w_0t = 0:.01:2*pi;
x =0;
for k = 1:2:n;
x = x + 1/k*sin(k*w_0t);
end
x=4/pi*x;
plot(w_0t, x), xlabel('w_0t, s')
text(3.5,.7,['Sum of ', num2str(n), ' sine waves'])
```



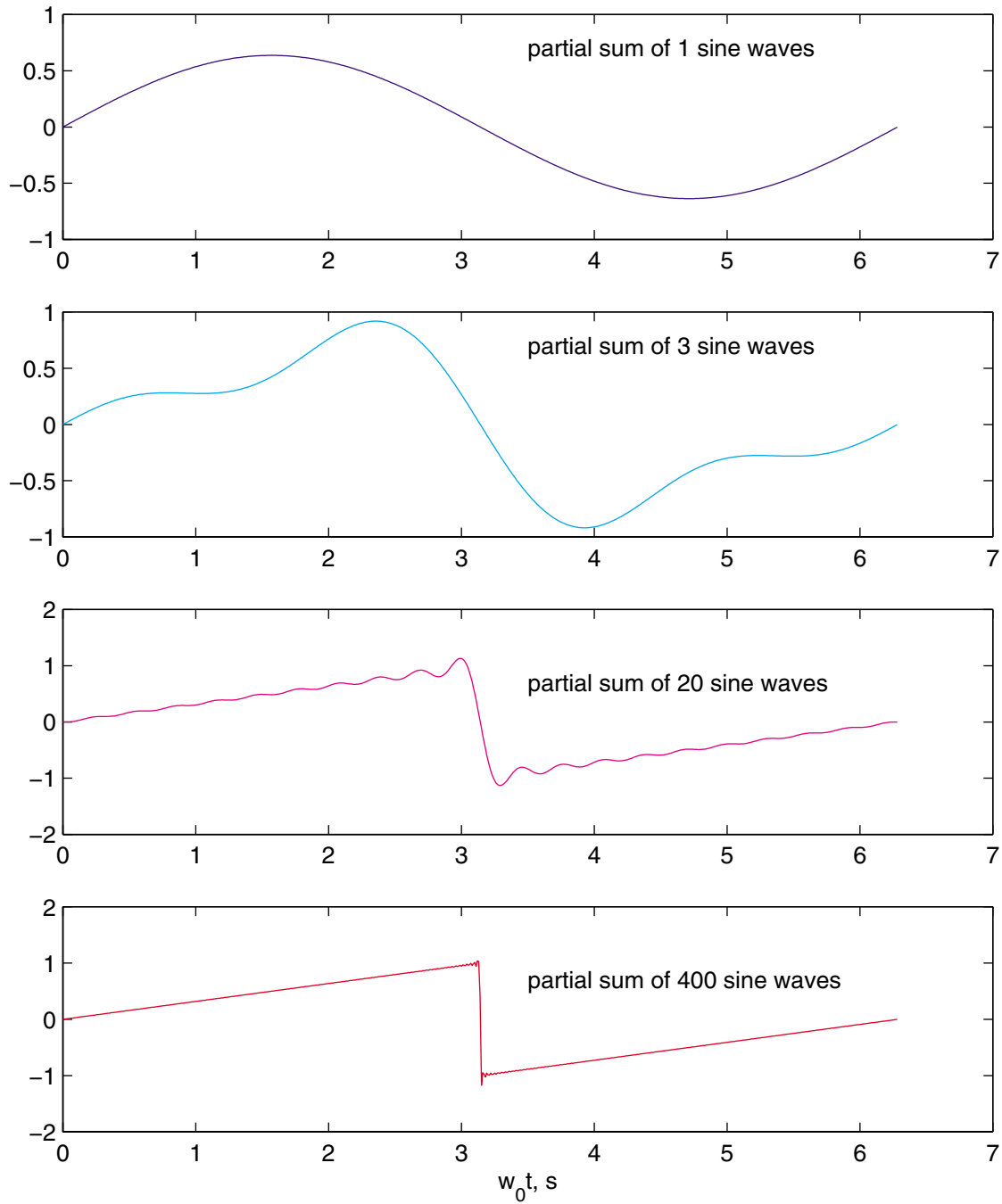
Example 3.2 (Example 3.2 Textbook)

As the second example consider the partial sum of the following harmonically related signals

$$\begin{aligned}x(t) &= \frac{2}{\pi} \left[\sin \omega_0 t - \frac{1}{2} \sin 2\omega_0 t + \frac{1}{3} \sin 3\omega_0 t - \dots \right] \\ &= \frac{2}{\pi} \sum_{n=1}^{\infty} -(-1)^n \frac{1}{n} \sin n\omega_0 t\end{aligned}$$

We write the following commands

```
%chs3_ex2.m
n=input('Enter the highest harmonics desired');
w_0t = 0:.01:2*pi;
x =0;
for k = 1:1:n;
x = x + -(-1)^k/k*sin(k*w_0t);
end
x=2/pi*x;
plot(w_0t, x), xlabel('w_0t, s')
text(3.5,.7,['Sum of ',num2str(n),' sine waves'])
```



Trigonometric form of Fourier Series

The trigonometric form of the Fourier series for a periodic signal $x(t)$ is given as the sum of sines and cosines of multiples of fundamental frequency ω_0

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t \quad (1)$$

The problem is to find a_0 , a_n , and b_n which are called the *Fourier series coefficients* for a given periodic signal. The constant term a_0 accounts for any dc offset in $x(t)$. To find a_0 , let us integrate both sides of the synthesis equation (1) over one period of the (T_0).

$$\int_{T_0} x(t) dt = a_0 \int_{T_0} dt + a_1 \int_{T_0} \cos \omega_0 t dt + a_2 \int_{T_0} \cos 2\omega_0 t dt + \dots \\ + b_1 \int_{T_0} \sin \omega_0 t dt + b_2 \int_{T_0} \sin 2\omega_0 t dt + \dots \quad (2)$$

Since the integral of sine and cosine over an integral number of periods are zero and the integral of the first term is $a_0 T_0$, we find

$$a_0 = \frac{1}{T_0} \int_{T_0} x(t) dt \quad (3)$$

To find a_n , let us multiply both side of (1) by $\cos m\omega_0 t$, and integrate both side of (1) over one period of the (T_0).

$$\int_{T_0} x(t) \cos m\omega_0 t dt = a_0 \int_{T_0} \cos m\omega_0 t dt + \int_{T_0} \left(\sum_{n=1}^{\infty} a_n \cos n\omega_0 t \right) \cos m\omega_0 t dt + \int_{T_0} \left(\sum_{n=1}^{\infty} b_n \sin n\omega_0 t \right) \cos m\omega_0 t dt \quad (4)$$

The first and the third terms on the right hand side are zero for all m , and n , and the second term is zero for $m \neq n$. Thus for $m = n$, the integral reduces to

$$\int_{T_0} x(t) \cos n\omega_0 t dt = a_n \int_{T_0} \cos^2 n\omega_0 t dt = \frac{a_n}{2} \int_{T_0} (1 - \cos 2n\omega_0 t) dt = \frac{a_n}{2} T_0 \quad (5)$$

or

$$a_n = \frac{2}{T_0} \int_{T_0} x(t) \cos n\omega_0 t dt \quad n \neq 0 \quad (6)$$

Similarly

$$b_n = \frac{2}{T_0} \int_{T_0} x(t) \sin n\omega_0 t dt \quad n \neq 0 \quad (7)$$

The Fourier series converges to the value of $x(t)$ at every point of continuity if $x(t)$ has continuous first and second derivatives. For a periodic function with discontinuity, the sufficient condition for pointwise convergence is given by *Dirichlet* condition

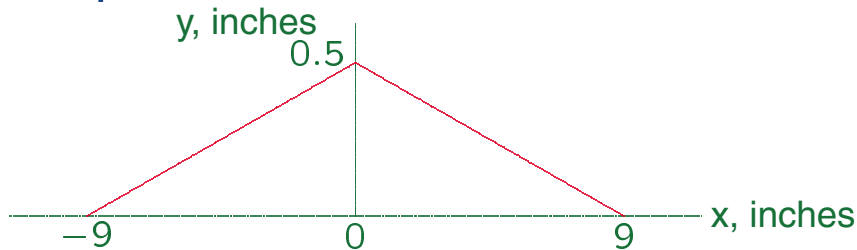
1. $x(t)$ is bounded.
2. $x(t)$ has a finite number of maxima and minima in one period.
3. $x(t)$ has a finite number of discontinuity in one period.

If $x(t)$ has discontinuity at a point, the Fourier series converges to the mean of the limits approached by $x(t)$ from the right and from the left.

For nonperiodic signals if $x(t)$ is given over a finite interval T_0 , the Fourier coefficients can be used for $x(t)$ within the interval. Outside the interval the Fourier series represent the periodic extension of $x(t)$.

Example 3.3 (Example 3.3 Textbook)

A violin String, to be plucked by a musician, has the shape shown



The Fourier series coefficient of this triangular wave are found to be

$$a_0 = \frac{A}{2}, \quad b_n = 0, \quad \text{for all } n$$

$$a_n = 0, \quad n \text{ even}, \quad a_n = \frac{4A}{\pi^2 n^2} \quad n \neq 0, \text{ \& odd}$$

Write the first four nonzero terms for the Fourier series for $|x| \leq 9$. The a_0 and the odd harmonics are

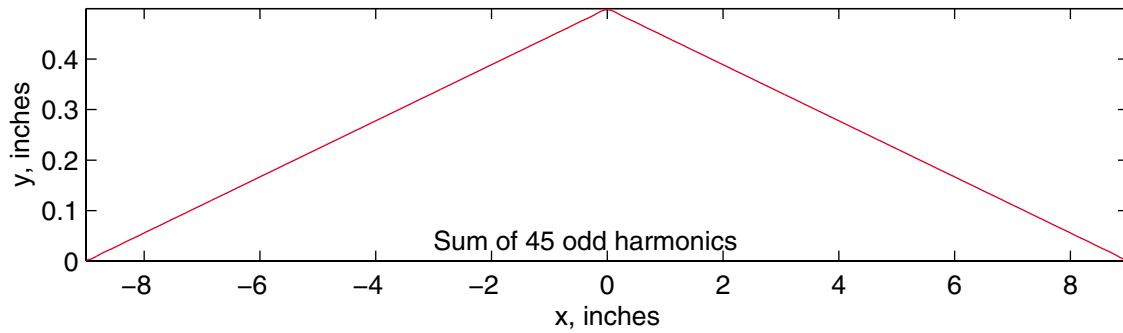
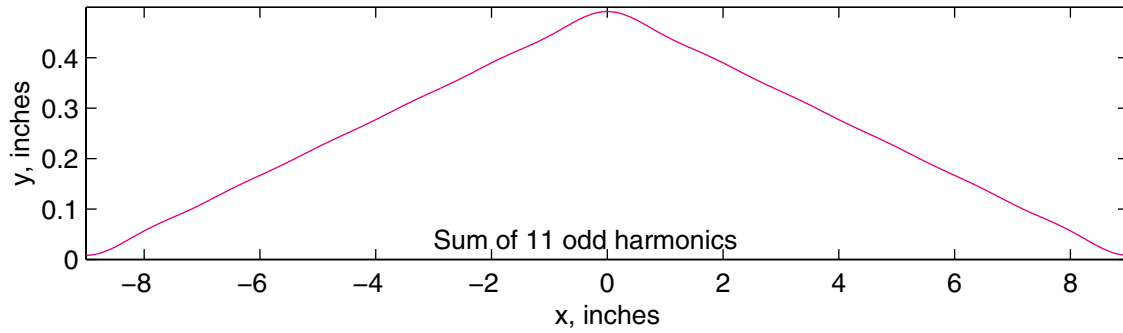
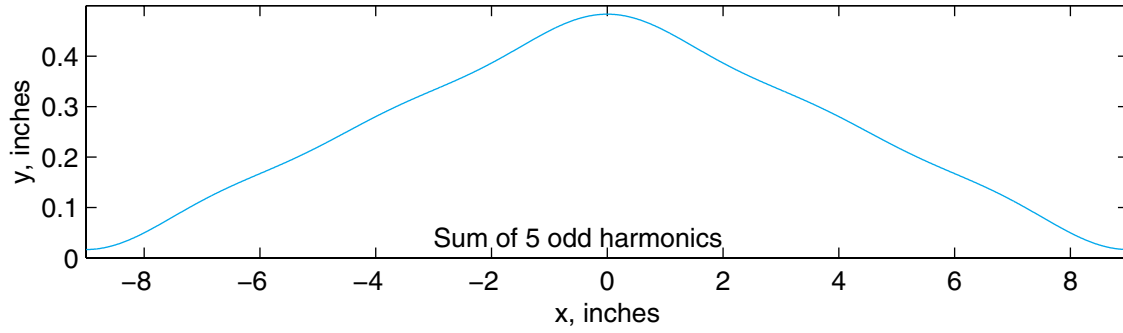
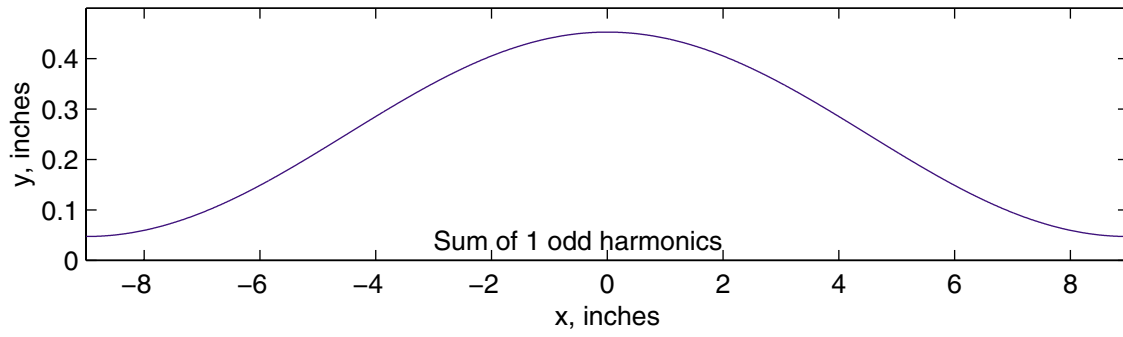
$$a_0 = \frac{0.5}{2} = \frac{1}{4} \quad a_1 = \frac{4(0.5)}{\pi^2} = \frac{2}{\pi^2}$$
$$a_3 = \frac{4(0.5)}{9\pi^2} = \frac{2}{9\pi^2} \quad a_5 = \frac{4(0.5)}{25\pi^2} = \frac{2}{25\pi^2}$$

Thus, for the interval $|x(t)| \leq 9$, we have

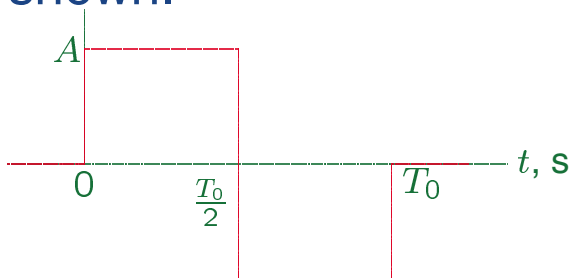
$$\hat{y}(x) = \frac{1}{4} + \frac{2}{\pi^2} \cos \frac{\pi x}{9} + \frac{2}{9\pi^2} \cos \frac{\pi x}{3} + \frac{2}{25\pi^2} \cos \frac{5\pi x}{9}$$

The following commands prompts the user for the highest odd harmonics and evaluates and plots $y(x)$ for $-9 \leq x \leq 9$.

```
%chs3_ex3.m
n = input('Enter the highest odd harmonics desired');
A = 0.5; T_0 = 18;
w_0 = 2*pi/T_0;
x = -9:.01:9;
y = 0;
for k = 1:2:n;
y = y + 4*A/(pi^2*k^2)*cos(k*w_0*x);
end
y = A/2 + y;
plot(x, y), xlabel('x, inches'), ylabel('y, inches')
axis([-9 9 0 0.5])
text(-3,0.04,['Sum of ',num2str(n),' odd harmonics'])
```



Example 3.4 (Example 3-4 Textbook) Obtain the trigonometric Fourier series for the periodic square signal shown.



Since the signal has a zero average value, $a_0 = 0$.

$$\begin{aligned}
 a_n &= \frac{2}{T_0} \int_0^{\frac{T_0}{2}} A \cos n\omega_0 t \, dt + \frac{2}{T_0} \int_{\frac{T_0}{2}}^{T_0} -A \cos n\omega_0 t \, dt \\
 &= \frac{2A}{T_0} \left[\frac{\sin n\omega_0 t}{n\omega_0} \Big|_0^{\frac{T_0}{2}} - \frac{\sin n\omega_0 t}{n\omega_0} \Big|_{\frac{T_0}{2}}^{T_0} \right] \\
 &= 0
 \end{aligned}$$

Note that the above expression resulted in zero, since

$$\omega_0 = \frac{2\pi}{T_0}$$

$$\begin{aligned}
b_n &= \frac{2}{T_0} \int_0^{\frac{T_0}{2}} A \sin n\omega_0 t \, dt + \frac{2}{T_0} \int_{\frac{T_0}{2}}^{T_0} -A \sin n\omega_0 t \, dt \\
&= \frac{2A}{T_0} \left[-\frac{\cos n\omega_0 t}{n\omega_0} \Big|_0^{\frac{T_0}{2}} + \frac{\cos n\omega_0 t}{n\omega_0} \Big|_{\frac{T_0}{2}}^{T_0} \right] \\
&= \frac{2A}{n\pi} (1 - \cos n\pi)
\end{aligned}$$

or

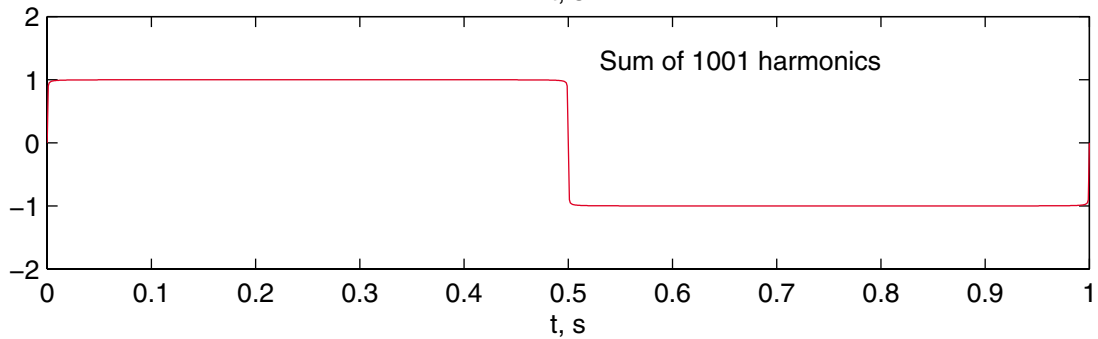
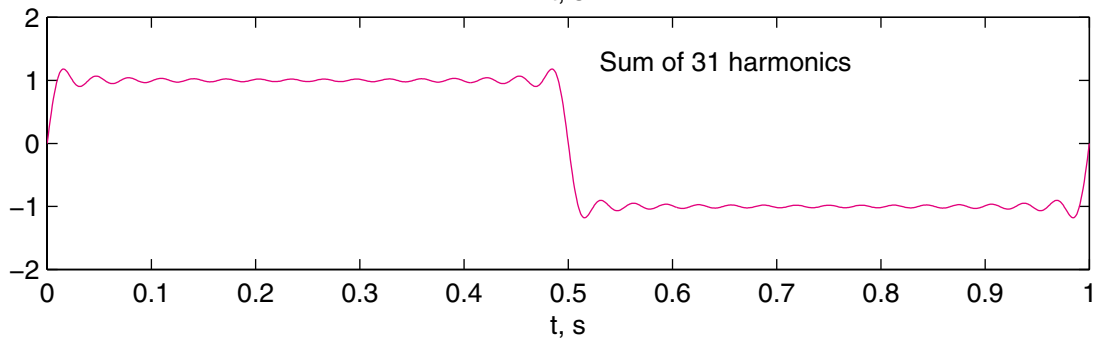
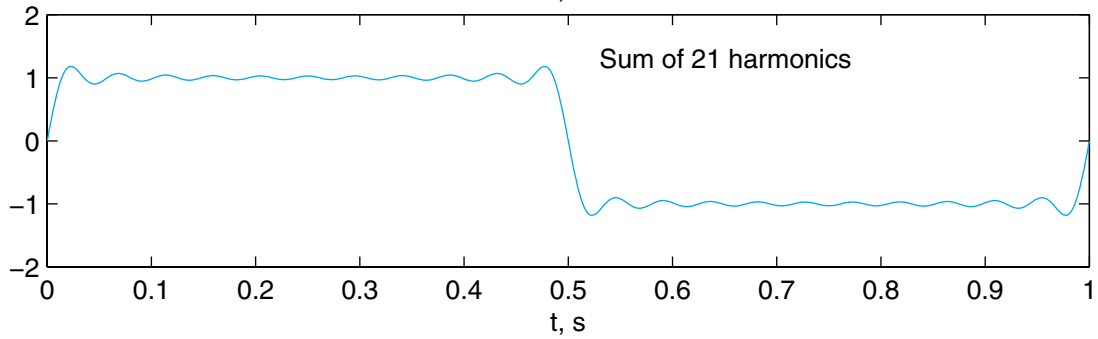
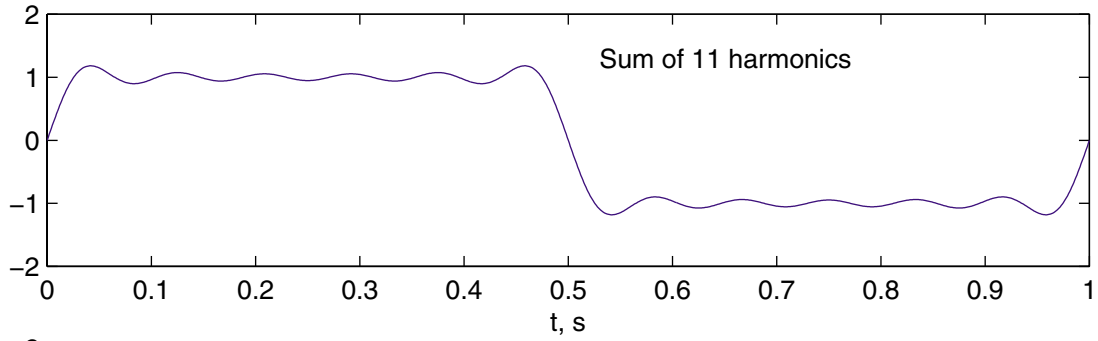
$$b_n = \begin{cases} \frac{4A}{n\pi}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

Thus, the Fourier series for the square signal of amplitude A with odd symmetry is

$$x(t) = \frac{4A}{\pi} \left(\sin \omega_0 t + \frac{1}{3} \sin 3\omega_0 t + \frac{1}{5} \sin 5\omega_0 t + \dots \right)$$

The following commands prompts the user for the highest odd harmonics and evaluates and plots $x(t)$ for $A = 1$, and $T_0 = 1$ second

```
%chs3_ex4.m
n = input('Enter the highest odd harmonics ');
A= 1; T_0 =1; w_0 = 2*pi/T_0;
t = 0:.001:1;
x=0;
for k = 1:2:n;
    x = x + 4*A/(k*pi)*sin(k*w_0*t);
end
plot(t, x), xlabel('t, s')
text(.5,1.3, ['Sum of ',num2str(n),' odd harmonics'])
```

Exponential form of Fourier series

Form the Euler's expression we have

$$\cos n\omega_0 t = \frac{1}{2}(e^{jn\omega_0 t} + e^{-jn\omega_0 t}) \quad \& \quad \sin n\omega_0 t = \frac{1}{2j}(e^{jn\omega_0 t} - e^{-jn\omega_0 t})$$

Substituting in the trigonometric Fourier series equation (1), we obtain

$$\begin{aligned} x(t) &= a_0 + \sum_{n=1}^{\infty} \frac{1}{2} a_n (e^{jn\omega_0 t} + e^{-jn\omega_0 t}) + \sum_{n=1}^{\infty} \frac{1}{2j} b_n (e^{jn\omega_0 t} - e^{-jn\omega_0 t}) \\ &= a_0 + \sum_{n=1}^{\infty} \frac{1}{2} (a_n - jb_n) e^{jn\omega_0 t} + \sum_{n=1}^{\infty} \frac{1}{2} (a_n + jb_n) e^{-jn\omega_0 t} \quad (8) \end{aligned}$$

or in terms of a new complex constants, the above expression becomes

$$\begin{aligned} x(t) &= \dots + X_{-2} e^{-j2\omega_0 t} + X_{-1} e^{-j\omega_0 t} \\ &\quad + X_0 + X_1 e^{j\omega_0 t} + X_2 e^{j2\omega_0 t} + \dots \end{aligned}$$

Thus, the exponential form of the Fourier series is given by

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t} \quad (9)$$

Now to find X_n , let us multiply both side of (9) by $e^{-jm\omega_0 t}$ and integrate over the period of $x(t)$. This result in

$$\begin{aligned} \int_{T_0} x(t) e^{-jm\omega_0 t} dt &= \int_{T_0} \left(\sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t} \right) e^{-jm\omega_0 t} dt \\ &= \sum_{n=-\infty}^{\infty} X_n \int_{T_0} e^{j(n-m)\omega_0 t} dt \end{aligned}$$

The term on the right hand side for $m \neq n$ represent the average sum of sine and cosine terms over a period which is zero. For $m = n$, we have

$$\int_{T_0} x(t) e^{-jn\omega_0 t} dt = T_0 X_n$$

Solving for X_n , we have

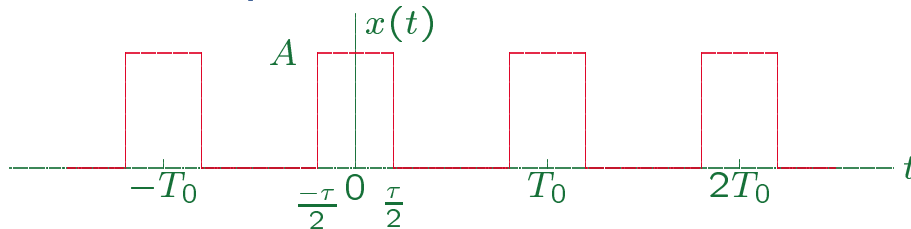
$$X_n = \frac{1}{T_0} \int_{T_0} x(t) e^{-jn\omega_0 t} dt \quad (10)$$

The exponential Fourier series coefficients were related to a_n , and b_n from

$$\begin{aligned} X_n &= \begin{cases} \frac{1}{2}(a_n - jb_n), & n > 0 \\ \frac{1}{2}(a_n + jb_n), & n < 0 \end{cases} \\ X_0 &= a_0 \end{aligned} \quad (11)$$

The computation of the coefficients of the exponential form is considerably easier than the trigonometric form. Furthermore, the exponential Fourier series provides us with a direct means of plotting two-sided amplitude and phase spectra of a signal.

Example 3.5 (Example 3-5 Textbook) For the next example we consider the train of rectangular pulses, known as *pulse train* as shown



This is an important signal, the clock signal in a digital computer is a pulse train. Also, the amplitude modulated signal in communication systems is a pulse train.

Find the coefficient of the complex exponential Fourier series for the pulse train signal shown above.

From (10), X_n is

$$\begin{aligned}
 X_n &= \frac{1}{T_0} \int_{-\tau/2}^{\tau/2} A e^{-jn\omega_0 t} dt & (12) \\
 &= \frac{-A}{jn\omega_0 T_0} e^{-jn\omega_0 t} \Big|_{-\tau/2}^{\tau/2} \\
 &= \frac{2A}{n\omega_0 T_0} \left[\frac{e^{jn\omega_0 \tau/2} - e^{-jn\omega_0 \tau/2}}{2j} \right] \quad n \neq 0
 \end{aligned}$$

or

$$X_n = \frac{2A}{n\omega_0 T_0} \sin n\omega_0 \tau / 2$$

Substituting for $\omega_0 = 2\pi f_0$, we get

$$X_n = \frac{A \sin \pi n f_0 \tau}{T_0 \pi n f_0}$$

Multiplying numerator and denominator by τ

$$X_n = \frac{A\tau \sin \pi n f_0 \tau}{T_0 \pi n f_0 \tau}$$

The signal $\text{sinc}(t)$, defined as

$$\text{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$

is encountered in many applications is known as Gaussian and Lorentzian function. Thus in terms of this signal X_n becomes

$$X_n = \frac{A\tau}{T_0} \text{sinc}(n f_0 \tau)$$

Note that from (12) for $n = 0$, $X_0 = \frac{A\tau}{T_0}$

Example 3.6 (Example 3-6 Textbook)

Find the coefficient of the complex exponential Fourier series for a half rectified sine signal, given by

$$x(t) = \begin{cases} A \sin \omega_0 t, & 0 \leq t \leq \frac{T_0}{2} \\ 0, & \frac{T_0}{2} \leq t \leq T_0 \end{cases}$$

From (10), X_n is

$$\begin{aligned} X_n &= \frac{1}{T_0} \int_0^{\frac{T_0}{2}} A \sin \omega_0 t e^{-jn\omega_0 t} dt \\ &= \frac{A}{2jT_0} \left[\int_0^{\frac{T_0}{2}} (e^{j\omega_0 t} - e^{-j\omega_0 t}) e^{-jn\omega_0 t} dt \right] \\ &= \frac{A}{2jT_0} \left[\int_0^{\frac{T_0}{2}} e^{j\omega_0(1-n)t} dt - \int_0^{\frac{T_0}{2}} e^{-j\omega_0(1+n)t} dt \right] \\ &= -\frac{A}{4\pi} \left[\frac{e^{j(1-n)\pi} - 1}{1-n} + \frac{e^{j(1+n)\pi} - 1}{1+n} \right] \quad n \neq \pm 1 \end{aligned}$$

Note that

$$e^{j(1\pm n)\pi} = \cos(1 \pm n)\pi + j \sin(1 \pm n)\pi = -(-1)^n$$

Therefore, we have

$$X_n = \begin{cases} 0, & n \text{ odd, } n \neq \pm 1 \\ \frac{A}{\pi} \frac{1}{1-n^2}, & n = 0, \pm 2, \pm 4, \dots (n = \text{even}) \\ \frac{A}{4jn}, & n = \pm 1 \quad (\text{see below}) \end{cases}$$

The case for $n = \pm 1$ was obtained separately as follows.

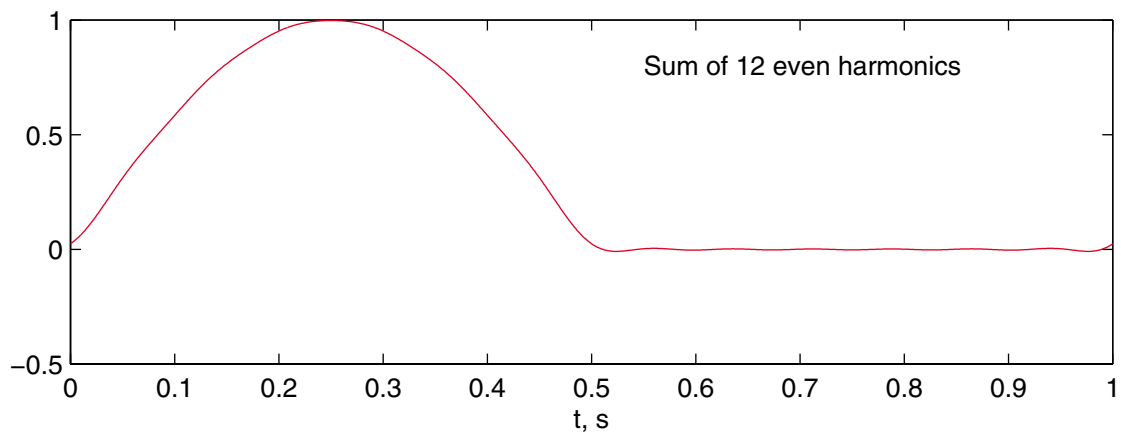
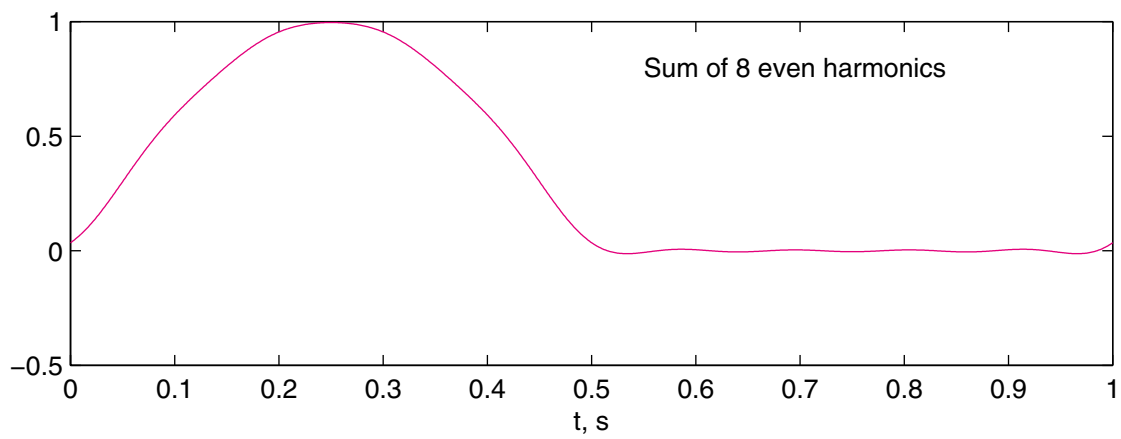
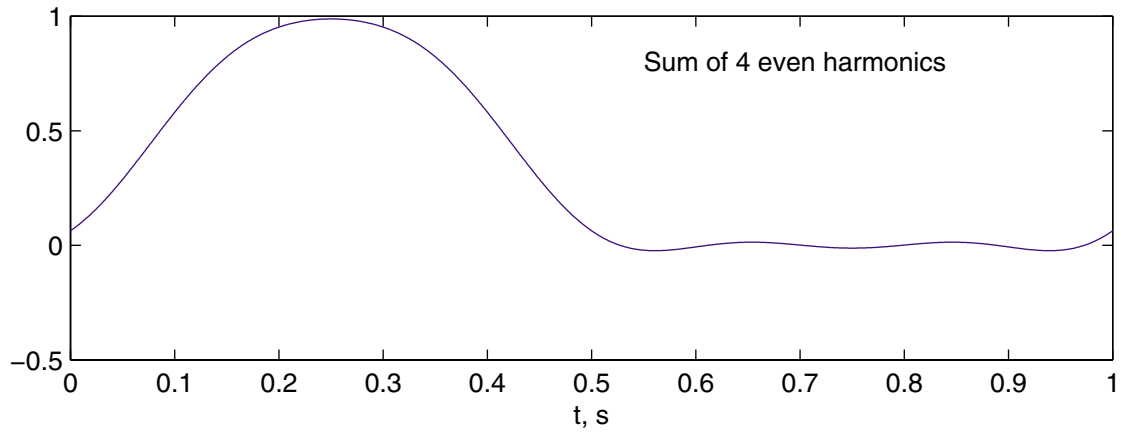
For $n = 1$, from (10), we have

$$\begin{aligned} X_1 &= \frac{A}{2jT_0} \int_0^{\frac{T_0}{2}} (e^{j\omega_0 t} - e^{-j\omega_0 t}) e^{-j\omega_0 t} dt \\ &= \frac{A}{2jT_0} \int_0^{\frac{T_0}{2}} (1 - e^{-j2\omega_0 t}) dt \\ &= \frac{A}{4j} \end{aligned}$$

Similarly, for $n = -1$, we find $X_1 = -\frac{A}{4j}$.

We use the following commands to evaluate the Fourier series for the half-rectified sine signal

```
%chs3_ex6.m
n = input('Enter the highest even harmonics ');
A= 1; T_0 =1; w_0 = 2*pi/T_0;
t = 0:.001:1;
m = [-n:2:-2, -1, 0, 1, 2:2:n];
p = length(m);
x= 0;
for k = 1: p
    if m(k) == 1 | m(k) == -1
        X_n = A/(4*m(k)*j);
    else
        X_n = A/(pi*(1-m(k)^2));
    end
x = x + X_n*exp(j*w_0*m(k)*t);
end
% we plot the real part of x to eliminate the
% small imaginary part due to computational error
plot(t, real(x)), xlabel('t, s')
text(.55,1.0,['Sum of ',num2str(n),' even harmonics'])
```



Half-wave Symmetry

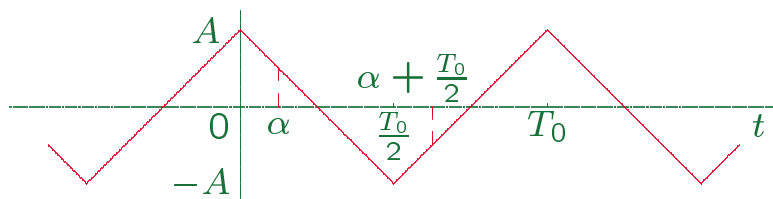
Half-wave symmetry is defined for periodic signals. A signal is half-wave even symmetrical if

$$x\left(t \pm \frac{T_0}{2}\right) = x(t)$$

A signal is half-wave odd symmetrical if

$$x\left(t \pm \frac{T_0}{2}\right) = -x(t)$$

For example, the triangular wave with the period T_0 as shown is a half-wave odd symmetrical signal, since the amplitude of the signal at any time $t = \alpha$ and at $t = \alpha \pm \frac{T_0}{2}$ differ only in sign.



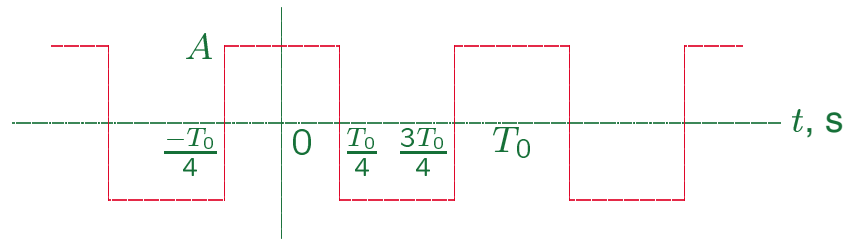
Simplification for Fourier series coefficient due to signal symmetry

The evaluation of the coefficients are simplified due to signal symmetry. For a real signal, we have the following properties

Signal Symmetry	Signal property	X_n coefficients	a_n & b_n coefficients
Even	$x(t) = x(-t)$	Real	$b_n = 0$, for all n
Odd	$x(t) = -x(-t)$	Imaginary	$a_n = 0$, for all n
Half-wave even	$x\left(t \pm \frac{T_0}{2}\right) = x(t)$	Complex $X_n = 0$, n odd	$a_n = b_n = 0$, n odd
Half-wave odd	$x\left(t \pm \frac{T_0}{2}\right) = -x(t)$	Complex $X_n = 0$, n even	$a_n = b_n = 0$, n even

Example 3.7

Obtain the trigonometric Fourier series for the even periodic square signal shown.



Since the signal has a zero average value, $a_0 = 0$. Also, since the signal is even, $b_n = 0$.

$$\begin{aligned} a_n &= \frac{2}{T_0} \int_{-\frac{T_0}{4}}^{\frac{T_0}{4}} A \cos n\omega_0 t \, dt + \frac{2}{T_0} \int_{\frac{T_0}{4}}^{\frac{3T_0}{4}} -A \cos n\omega_0 t \, dt \\ &= \frac{2A}{T_0} \left[\frac{\sin n\omega_0 t}{n\omega_0} \Big|_{-\frac{T_0}{4}}^{\frac{T_0}{4}} - \frac{\sin n\omega_0 t}{n\omega_0} \Big|_{\frac{T_0}{4}}^{\frac{3T_0}{4}} \right] \\ &= \frac{4A}{n\pi} \sin \frac{n\pi}{2} \end{aligned}$$

Exponential coefficient $X_n = \frac{1}{2}(a_n - jb_n) = \frac{1}{2}a_n$, or

$$X_n = \begin{cases} \frac{2A}{|n|\pi}, & n = \pm 1, \pm 5, \dots \\ \frac{-2A}{|n|\pi}, & n = \pm 3, \pm 7, \dots \\ 0, & n \text{ even} \end{cases}$$

Thus, the Fourier series for the even square signal is

$$x(t) = \frac{4A}{\pi} \left(\cos \omega_0 t - \frac{1}{3} \cos 3\omega_0 t + \frac{1}{5} \cos 5\omega_0 t - \dots \right)$$

Example 3.8 (3-7 Textbook)

Find the exponential Fourier series for an odd periodic square signal.

Consider the even periodic square signal of Example 3.7. Clearly, a time shift of $\tau_0 = \frac{T_0}{4}$ will result in an odd periodic square signal. The exponential FS coefficients of a signal $x(t)$ is

$$X_n = \frac{1}{T_0} \int_{T_0} x(t) e^{-jn\omega_0 t} dt$$

If the signal goes through a time-shift of τ_0 , then its exponential FS coefficients denoted by Y_n is

$$Y_n = \frac{1}{T_0} \int_{T_0} x(t - \tau_0) e^{-jn\omega_0 t} dt$$

With a change of variable in the integral, we find

$$Y_n = X_n e^{-j\frac{2n\pi\tau_0}{T_0}}$$

Thus, exponential FS for the odd periodic signal in terms of the even square periodic signal coefficients using $\tau_0 = \frac{T_0}{4}$, becomes

$$X_{n,odd} = \left(e^{-j\frac{n\pi}{2}} \right) X_{n,even} = \left(-j \sin \frac{n\pi}{2} \right) X_{n,even} \quad n \text{ odd}$$

where $X_{n,even}$ was found in Example 3.7.

In Example 3.7 for an even periodic square signal, we found

$$X_n = \frac{1}{2}a_n = \frac{2A}{n\pi} \sin \frac{n\pi}{2}$$

Thus, from the result of Example 3.8, for an odd periodic square signal, the complex Fourier coefficients is given by

$$\begin{aligned} X_n &= \left(-j \sin \frac{n\pi}{2} \right) \frac{2A}{n\pi} \sin \frac{n\pi}{2} \\ &= -j \frac{A}{n\pi} (1 - \cos n\pi) \end{aligned}$$

In Example 3.4, for the odd periodic square signal we found $a_n = 0$ and $b_n = \frac{2A}{n\pi}(1 - \cos n\pi)$, which will result in the same complex coefficients, using $X_n = \frac{1}{2}(a_n - jb_n)$.

Parseval's Theorem

The average normalized power of a periodic signal $x(t)$ is given by

$$P_{av} = \frac{1}{T_0} \int_{T_0} |x^2(t)| dt = \frac{1}{T_0} \int_{T_0} x(t)x^*(t) dt$$

Replace $x^*(t)$ by its Fourier's series expression

$$\begin{aligned} P_{av} &= \frac{1}{T_0} \int_{T_0} x(t) \left(\sum_{n=-\infty}^{\infty} X_n^* e^{-jn\omega_0 t} \right) dt \\ &= \sum_{n=-\infty}^{\infty} X_n^* \left(\frac{1}{T_0} \int_{T_0} x(t) e^{-jn\omega_0 t} dt \right) \\ &= \sum_{n=-\infty}^{\infty} X_n^* X_n = \sum_{n=-\infty}^{\infty} |X_n|^2 \\ &= X_0^2 + 2 \sum_{n=1}^{\infty} |X_n|^2 \end{aligned}$$

Therefore, the average power of a periodic signal can be found by its Fourier's series components, namely its frequency spectrum $X[n]$.

Example 3.9 (Example 3.8 Textbook)

Consider the sine wave $x(t) = 4\sin 50\pi t$. Find the average power of the signal using $x(t)$, and from its two-sided frequency spectrum.

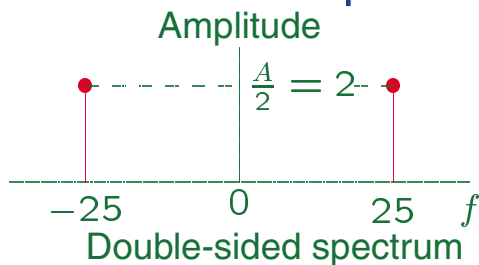
$$f_0 = \frac{50}{2} = 25 \text{ Hz} \quad \Rightarrow \quad T_0 = \frac{1}{25} = 0.04 \text{ second}$$

$$P = \frac{1}{T_0} \int_0^{T_0} |x(t)|^2 dt = \frac{1}{0.04} \int_0^{0.04} 16 \sin^2 50\pi t dt = \frac{16}{2} = 8 \text{ W}$$

To find the frequency spectrum we transform $x(t)$ into cosine signal, i.e,

$$x(t) = 4 \sin 50\pi t = 4 \cos\left(50\pi t - \frac{\pi}{2}\right)$$

The double-sided spectrum is as shown.



From Parseval's theorem, the average power is

$$P = (2)^2 + (2)^2 = 8 \text{ W}$$

Line Spectra

We have seen that a cosine signal can be represented by the sum of two counter rotating signals, These phasors are characterized in the frequency domain by their amplitudes versus frequency and their phase angle versus frequency, known as two-sided amplitude spectrum and phase spectrum.

Since each Fourier's series coefficient of a signal is associated with a complex sinusoid of a harmonically related frequency, these coefficients can conveniently be presented in the frequency domain by their amplitude and phase spectra. These discrete signals are called *line spectra*. Plot of the exponential form X_n , and θ_n for all $\pm n$ are called the **two-sided line spectra**.

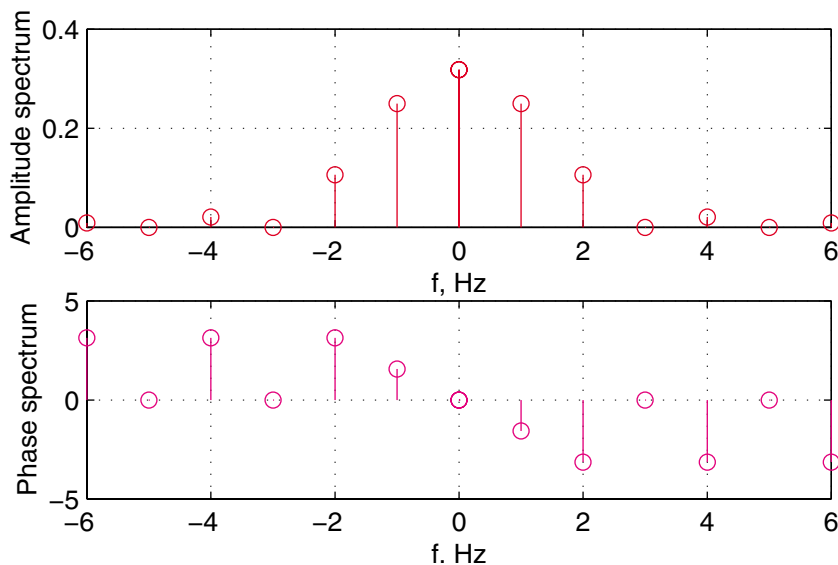
Example 3.10

Plot the two-sided line spectra for the Fourier's series coefficients of the half-wave rectifier of Example 3.6.

The complex exponential Fourier's series of the half-wave rectifier were found as

$$X_n = \begin{cases} 0, & n \text{ odd}, n \neq \pm 1 \\ \frac{A}{\pi} \frac{1}{1-n^2}, & n = 0, \pm 2, \pm 4, \dots (n = \text{even}) \\ \frac{A}{4jn}, & n = \pm 1 \end{cases}$$

Evaluating the above coefficients for $A = 1$, and $T_0 = 1$. $n = 0, \pm 1, \pm 2, \pm 4, \pm 6$, results in the line spectra shown.



MATLAB statements for plotting the line spectra of Example 3.10.

```
%chs3_ex10.m
A= 1; T_0 =1; f_0 = 1/T_0;
nf=(0:1:6)*f_0;
m=length(nf);
X = zeros(1,m)+j*zeros(1,m);%Set X for all nf to 0
X(1) = A/pi; % X for nf = 0
X(2) = A/(-j*4); % X for nf = 1
for k=3:2:m % X for even nf
    X(k) = A/(pi*(1-nf(k)^2));
end
Xf = abs(X); Xtheta = -angle(X);
nf_2 = [-nf, nf];
Xf_2 = [Xf, Xf]; % 2-sided amplitude spectrum
Xtheta_2=[-Xtheta, Xtheta];% -sided angle spectrum
subplot(2,1,1), stem(nf_2, Xf_2, 'r'), grid,
xlabel('f, Hz'), ylabel('Amplitude spectrum')
subplot(2,1,2), stem(nf_2, Xtheta_2, 'm'), grid,
xlabel('f, Hz'), ylabel('Phase spectrum')
```

Example 3.11

Plot the two-sided line spectra for the Fourier's series coefficients of a pulse train with time delay t_0 .

In Example 3.5, the complex exponential Fourier's coefficients for the pulse train were found to be

$$X_n = \frac{A\tau}{T_0} \text{sinc}(nf_0\tau)$$

If the pulse train is time-shifted by t_0 , the coefficients become (See the relation $Y_n = X_n e^{-j2n\pi f_0 t_0}$ found in Example 3.8)

$$X_n = \frac{A\tau}{T_0} \text{sinc}(nf_0\tau) e^{-j2n\pi n f_0 t_0}$$

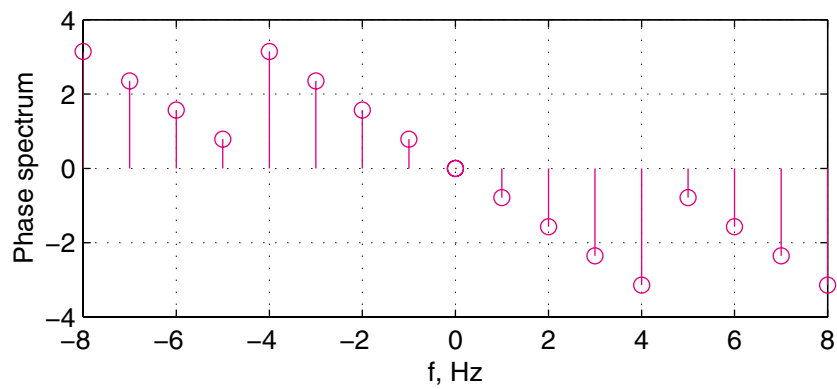
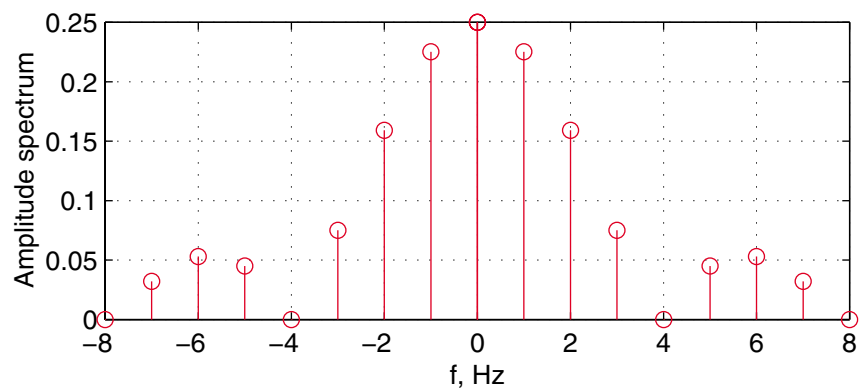
The file `ch3_ex11.m` evaluates X_n , and plots the line spectra for the pulse train. The user is prompted to enter the pulse amplitude A , the pulse width τ , period T_0 , and the time delay t_0 . Running `ch3_ex11` for $A = 1$, $\tau = 0.25$, $T_0 = 1$; and $t_0 = \tau/2$, results in

Amplitude $A = 1$

Width $\tau = .25$

Period $T_0 = 1$

Time delay $t_0 = .25/2$



MATLAB statements for plotting the line spectra of Example 3.11.

```
%chs3_ex11.m
disp('Line Spectra for a pulse train:')
disp('Amplitude A, width tau, period T_0, & time delay t_0')
A = input('Amplitude A = ');
tau = input('Width tau = ');
T_0 = input('Period = ');
t_0 = input('Time delay t_0 = ');
f_0 = 1/T_0;
n=(0:1:8*T_0);
X = A*tau/T_0*sinc(n*f_0*tau).*exp(-j*2*pi*n*f_0*t_0);
Xf = abs(X); Xtheta = angle(X);
nf_2 = [-n*f_0, n*f_0];
Xf_2 = [Xf, Xf]; % 2-sided amplitude spectrum
Xtheta_2 = [-Xtheta, Xtheta]; % 2-sided angle spectrum
subplot(2,1,1), stem(nf_2, Xf_2, 'r'), grid,
xlabel('f, Hz'), ylabel('Amplitude spectrum')
subplot(2,1,2), stem(nf_2, Xtheta_2, 'm'), grid,
xlabel('f, Hz'), ylabel('Phase spectrum')
```

Steady-state response of a LTI system to a periodic input

We have seen that the response of a LTI system to a sinusoidal signal of frequency ω rad/sec is a sinusoid of the same frequency, but different amplitude and phase dictated by the frequency response transfer function, i.e. for the sinusoidal signal $x_1(t) = A \cos(\omega t)$, the steady-state sinusoidal signal is

$$y_{ss}(t) = A|H(\omega)| \cos(\omega t + \theta(\omega))$$

or in terms of the rotating phasor $Ae^{j\omega t}$, we have

$$y_{ss}(t) = AH(\omega)e^{j\omega t}$$

In the previous chapter we found $H(\omega)$, as

$$H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t} dt$$

We shall see that the above integral is the *Fourier transform* of the impulse response.

Using the superposition, we can extend the steady-state response to a periodic signal with complex Fourier coefficients X_n , as

$$y(t) = \sum_{n=-\infty}^{\infty} X_n H(n\omega_0) e^{jn\omega_0 t}$$

In polar form $H(\omega) = A(\omega)e^{j\theta(\omega)}$, the steady-state response to a periodic signal represented by its Fourier coefficients X_n becomes

$$y(t) = X_0 H(0) + \sum_{n=1}^{\infty} 2|X_n| A(n\omega_0) \cos[n\omega_0 t + \angle X_n + \theta(n\omega_0)]$$

Example 3.12 (Example 3-10, 3-11 Textbook)

An odd rectangular signal represented by its complex Fourier coefficients is applied to the low pass RC filter. Obtain the output response, plot the input and output signals, and their amplitude phase spectrum.

The Frequency Response Transfer Function of the low pass RC circuit is

$$H(\omega) = \frac{1}{1 + j\omega RC}$$

and from Example 3.8, the Complex Fourier coefficients are given by

$$X_n = -j \frac{A}{n\pi} (1 - \cos n\pi);$$

The response spectrum is

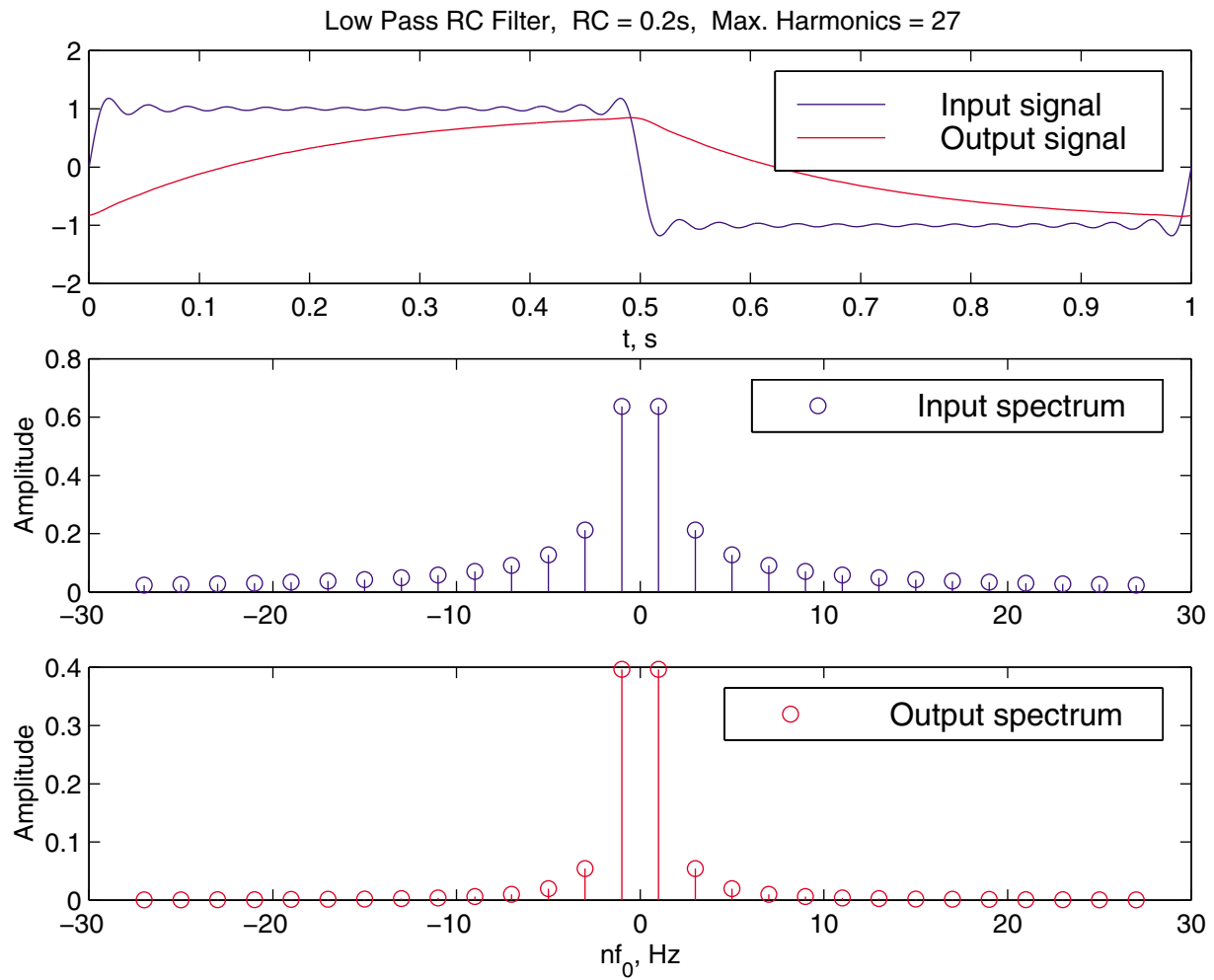
$$Y_n = H(n\omega_0) X_n$$

and the output response is

$$y(t) = \sum_{n=-\infty}^{\infty} X_n H(n\omega_0) e^{jn\omega_0 t}$$

We use the following commands

```
RC = input('Enter the filter time constant RC = ');
n = input('Enter the highest odd harmonic n = ');
A =1; T_0 = 1; , f_0 = 1/T_0;
t=0:0.001:1;
w_0 = 2*pi*f_0;
x = 0; y=0;
m =0; H=0; X_n=0; Y_n=0;
for k = -n:2:n
    m=m+1;
    H(m) = 1/(1 + j*w_0*k*RC);
    X_n(m) = -j*A/(k*pi)*(1 - cos(k*pi));
    Y_n(m) = H(m)*X_n(m);
    x = x + X_n(m)*exp(j*k*w_0*t);
    y = y + Y_n(m)*exp(j*k*w_0*t);
end
nf_0 = [-n:2:n]*f_0;
subplot(3,1,1), plot(t, real(x), 'b', t, real(y), 'r')
xlabel('t, s')
title(['Low Pass RC Filter, RC = ', num2str(RC),...
's, Max. Harmonics = ', num2str(n)])
legend('Input signal', 'Output signal')
subplot(3,1,2), stem(nf_0, abs(X_n), 'b')
ylabel('Amplitude'), legend('Input spectrum')
subplot(3,1,3), stem(nf_0, abs(Y_n), 'r')
xlabel('nf_0, Hz'), ylabel('Amplitude')
legend('Output spectrum')
```



Distortionless Systems

A continuous LTI system described in terms of its frequency response is said to have *distortionless transmission* if the output signal is an exact replica of the input signal, except for

- A possible scaling of the amplitude
- A constant time delay

The frequency response transfer function of a distortionless system is

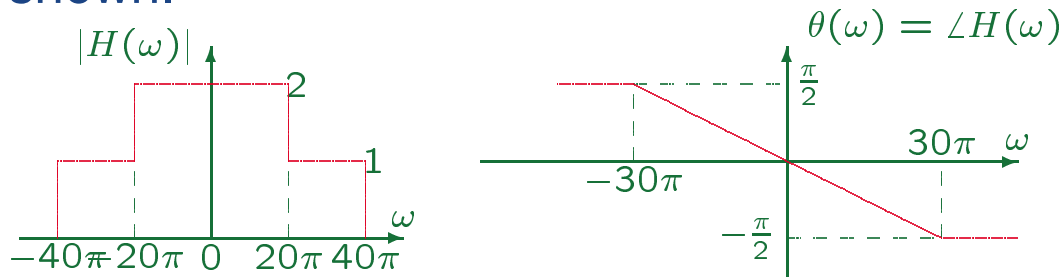
$$H_d(\omega) = K e^{-j\tau_0\omega}$$

Therefore, the amplitude response is the same for all the frequency components of the input, and phase response is a linear function of frequency. The signal $x(t)$ is transmitted through the system without distortion, and the output is given by

$$y(t) = Kx(t - \tau_0)$$

where K is the scaling factor, and τ_0 is the time delay.

Example 3.13 (Example 3-12 Textbook) An system has idealized frequency response transfer function as shown.



Determine the response for the following input

1. $x_1(t) = 2 \cos 10\pi t + \sin 12\pi t$
2. $x_2(t) = 2 \cos 10\pi t + \sin 26\pi t$
3. $x_3(t) = 2 \cos 26\pi t + \sin 34\pi t$

From the FRTF response, we have

$$|H(\omega)| = \begin{cases} 1 & -40\pi \leq \omega \leq -20\pi \\ 2 & -20\pi \leq \omega \leq 20\pi \\ 1 & 20\pi \leq \omega \leq 40\pi \end{cases}$$

$$\theta(\omega) = \begin{cases} \frac{\pi}{2} & \omega \leq -30\pi \\ -\frac{1}{60}\omega & -30\pi \leq \omega \leq 30\pi \\ -\frac{\pi}{2} & \omega \geq 30\pi \end{cases}$$

$\theta(\omega)$ at frequencies of the input signals are

$$\begin{aligned}\theta(10\pi) &= -\frac{1}{60}(10\pi) = -\frac{\pi}{6}, & \theta(12\pi) &= -\frac{\pi}{5} \\ \theta(26\pi) &= -\frac{1}{60}(26\pi) = -\frac{13\pi}{30}, & \theta(34\pi) &= -\frac{\pi}{2}\end{aligned}$$

The steady-state response for the given inputs are

- $$\begin{aligned}y_1(t) &= 4 \cos(10\pi t - \frac{\pi}{6}) + 2 \sin(12\pi t - \frac{\pi}{5}) \\ &= 4 \cos 10\pi(t - \frac{1}{60}) + 2 \sin 12\pi(t - \frac{1}{60})\end{aligned}$$
- $$\begin{aligned}y_2(t) &= 4 \cos(10\pi t - \frac{\pi}{6}) + \sin(26\pi t - \frac{13\pi}{30}) \\ &= 4 \cos 10\pi(t - \frac{1}{60}) + \sin 26\pi(t - \frac{1}{60})\end{aligned}$$
- $$\begin{aligned}y_3(t) &= 2 \cos(26\pi t - \frac{13\pi}{30}) + \sin(34\pi t - \frac{\pi}{2}) \\ &= 2 \cos 26\pi(t - \frac{1}{60}) + \sin 34\pi(t - \frac{1}{68})\end{aligned}$$

For a distortionless system, $y(t) = Kx(t - \tau_0)$. Therefore, we can see that only input $x_1(t)$ is passed without distortion. The system produces amplitude distortion on $x_2(t)$, and phase distortion on $x_3(t)$.

Exercise (p.133) The frequency response transfer function of the second-order low-pass filter is given by

$$H(\omega) = \frac{\omega_c^2}{\omega_c^2 - \omega^2 + j\sqrt{2}\omega_c\omega}$$

(a) We find the amplitude and the phase angle to be

$$A(\omega) = \left[1 + \left(\frac{\omega}{\omega_c} \right)^4 \right]^{-\frac{1}{2}}$$

$$\theta(\omega) = \begin{cases} -\tan^{-1} \frac{\sqrt{2}(\omega/\omega_c)}{1-\omega^2/\omega_c^2} & \omega < \omega_c \\ -\pi + \tan^{-1} \frac{\sqrt{2}(\omega/\omega_c)}{\omega^2/\omega_c^2-1} & \omega > \omega_c \end{cases}$$

(b) $\omega_c = 600\pi$ r/s and input is a square signal with FS given by (3-25) with $\omega_0 = 200\pi$ r/s. Obtain the attenuation and phase shifts introduced by the filter in the first 3 terms of the FS.

The FS of the square signal contains odd harmonics, and $|H(\omega)|$, and $\theta(\omega)$ for the first 3 harmonics are

$$|H(200\pi)| = \left[1 + \left(\frac{200\pi}{600\pi} \right)^4 \right]^{-\frac{1}{2}} = 0.994$$

$$|H(3 \times 200\pi)| = \left[1 + \left(\frac{3 \times 200\pi}{600\pi} \right)^4 \right]^{-\frac{1}{2}} = 0.707$$

$$|H(5 \times 200\pi)| = \left[1 + \left(\frac{5 \times 200\pi}{600\pi} \right)^4 \right]^{-\frac{1}{2}} = 0.339$$

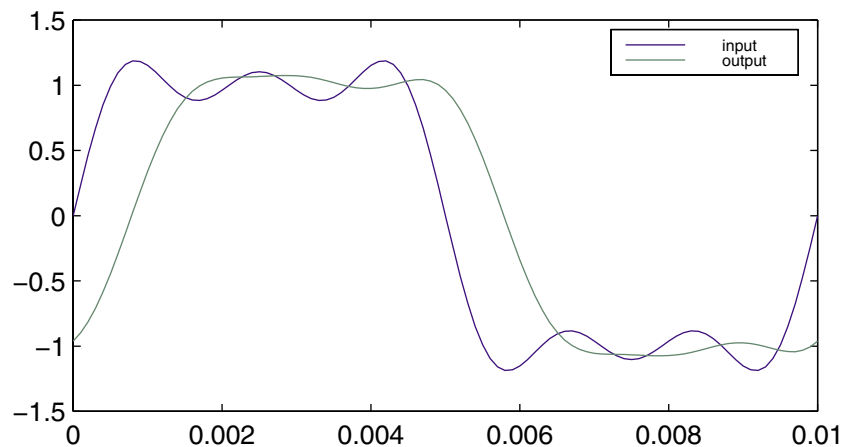
$$\theta(200\pi) = -\tan^{-1} \frac{\sqrt{2} \frac{200\pi}{600\pi}}{1 - \left(\frac{200\pi}{600\pi}\right)^2} = -27.9^\circ$$

$$\theta(600\pi) = -\tan^{-1} \infty = -90^\circ$$

$$\theta(1000\pi) = -\pi + \tan^{-1} \frac{\sqrt{2} \frac{1000\pi}{600\pi}}{\left(\frac{1000\pi}{600\pi}\right)^2 - 1} = -127^\circ$$

We use the following commands to plot $x(t)$ and $y(t)$.

```
A = 1; omega_0 = 200*pi;
t=0:.0001:.01;
x = 4*A/pi*(sin(omega_0*t) + 1/3*sin(3*omega_0*t)+...
1/5*sin(5*omega_0*t));
y=4*A/pi*(0.994*sin(omega_0*t-27.9*pi/180)+1/3*0.707*sin...
(3*omega_0*t-pi/2)+1/5*0.339*sin(5*omega_0*t-127*pi/180));
plot(t,x , t, y)
legend('input', 'output')
```



Rate of Convergence of Fourier Spectra

The Fourier series coefficients X_n for a periodic function $x(t)$ always decrease in magnitude with the increase in the harmonic index n . It can be shown that for sufficiently large n , the Fourier coefficients approach zero with a convergence rate of $1/n^k$, where k is the number of times $x(t)$ is differentiated before the appearance of an impulse in the derivative. A large k implies a faster convergence and fewer terms in the truncated Fourier series. For example, the square and sawtooth waves produce impulse at the first derivative, and have convergence rate of $1/n$. Whereas, the half-rectified sine, and trapezoidal signals need two derivatives before producing impulses, and thus have a convergence rate of $1/n^2$.

Example 3.14

One pulse of a periodic signal is defined by

$$x(t) = \sum_{n=-\infty}^{\infty} x_p(t - 2n)$$

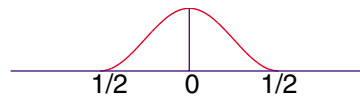
where

$$x_p(t) = \cos^2 \pi t = \frac{1}{2}(1 + \cos 2\pi t) \quad -\frac{1}{2} \leq t \leq \frac{1}{2}$$

Find derivatives of $x_p(t)$, until it would yield an impulse.

$x_p(t)$, up to its third derivatives are shown below

$$x = 1/2(1 + \cos 2\pi t)$$



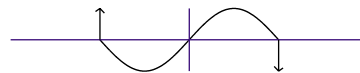
$$dx/dt = -\pi \sin 2\pi t$$



$$d^2x/dt^2 = -2\pi^2 \cos 2\pi t$$



$$d^3x/dt^3 = 2\pi^2 \delta(t+1/2) + 4\pi^3 \sin 2\pi t - 2\pi^2 \delta(t-1/2)$$



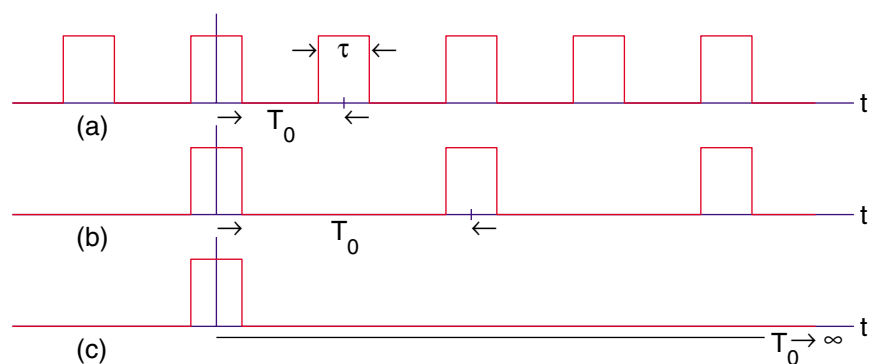
The impulse appears at the third derivative. Thus, the Fourier coefficients have a convergence rate of $1/n^3$.

Fourier Transform

In the previous chapter, we expressed a periodic signal as a weighted superposition of harmonically related sinusoidal signals. However, many signals in engineering are aperiodic and signals of finite duration. We obtain the Fourier transform by first considering the Fourier series for periodic signals, which was defined in exponential form as

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn2\pi f_0 t}$$
$$X_n = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t) e^{-jn2\pi f_0 t} dt$$

Consider the pulse train signal of width τ , and period T_0 shown below, and see what happens as period gets longer and longer and finally grows to infinity as shown in Figures (b) and (c).



The signal in (c) is never repeated and can be considered as a pulse signal of finite duration. As $T_0 \rightarrow \infty$, the spacing between harmonics goes to a differential and becomes infinitesimally small, and $f_0 = \frac{1}{T_0}$ approaches df , nf_0 becomes a continuous variable f , and the summation in the Fourier series becomes integral, i.e.,

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df \quad (1)$$

We express X_n in the form

$$T_0 X_n = \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)e^{-jn2\pi f_0 t} dt$$

As $T_0 \rightarrow \infty$ and $X_n \rightarrow 0$, the spectrum becomes continuous and $\lim_{T_0 \rightarrow \infty} T_0 X_n$ is shown by $X(f)$

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft} dt \quad (2)$$

Equation (2) defines the *Fourier transform*, and Equation (1) the *inverse Fourier transform* which applies to continuous aperiodic signals. They are written as

$$\mathcal{F}\{x(t)\} = X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft} dt$$

$$\mathcal{F}^{-1}\{X(f)\} = x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft} df$$

Together they are called a *transform pair* and are represented in short hand notation as

$$x(t) \xleftrightarrow{\mathcal{F}} X(f)$$

Example 4.1 (Example 4-1 Textbook)

Determine the Fourier transform for

(a) The even rectangular pulse $x_a(t) = \Pi(\frac{1}{\tau}t)$

(b) The odd rectangular pulse

$$x_b(t) = \Pi\left(\frac{2t}{\tau} + \frac{1}{2}\right) - \Pi\left(\frac{2t}{\tau} - \frac{1}{2}\right)$$

(a) Since $\Pi(\frac{1}{\tau}t)$ is zero outside $t = \pm\frac{\tau}{2}$, the Fourier transform of $x_a(t)$ becomes

$$\begin{aligned} X_a(f) &= \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} e^{-j2\pi ft} dt = \frac{e^{j\pi\tau f} - e^{-j\pi\tau f}}{j2\pi f} = \frac{\sin\pi\tau f}{\pi f} \\ &= \tau \frac{\sin(\pi\tau f)}{\pi\tau f} = \tau \operatorname{sinc}(\tau f) \end{aligned}$$

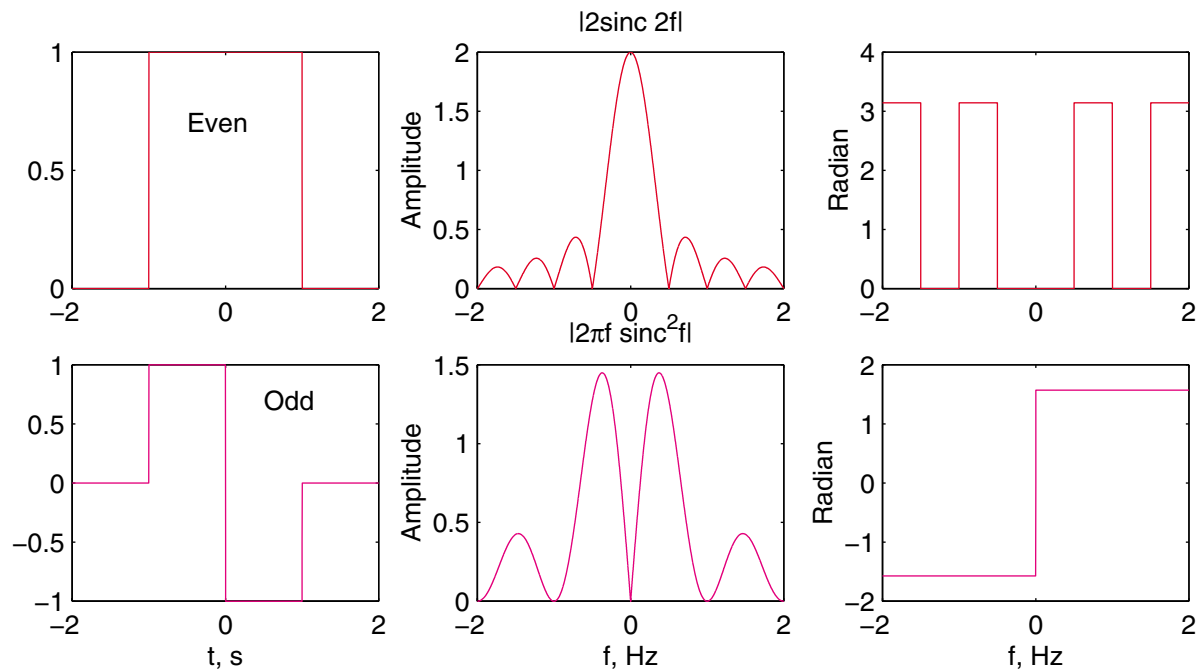
(b)

$$\begin{aligned} X_b(f) &= \int_{-\frac{\tau}{2}}^0 e^{-j2\pi ft} dt - \int_0^{\frac{\tau}{2}} e^{-j2\pi ft} dt \\ &= \frac{-1}{j2\pi f} [2 - (e^{j\pi\tau f} + e^{-j\pi\tau f})] = \frac{j}{\pi f} [1 - \cos\pi\tau f] \\ &= j \frac{\tau^2}{2} \pi f \frac{\sin^2(\frac{\pi\tau}{2}f)}{(\frac{\pi\tau}{2}f)^2} = j \frac{\tau^2}{2} \pi f \operatorname{sinc}^2\left(\frac{\tau}{2}f\right) \end{aligned}$$

We use the following commands to plot the even and odd rectangular signals (Amplitude 1 and $\tau = 2$), and the amplitude and phase angle of their Fourier transforms.

```
t=-2:.001:2;
tau=2;
x_a = pls_fn(t/tau);
f = -2:.001:2;
X_af = tau*sinc(tau*f);
subplot(4,3,1), plot(t, x_a, 'r')
text(-.5, .7, 'Even')
subplot(4,3,2), plot(f, abs(X_af), 'r')
title('|2sinc 2f|'), ylabel('Amplitude')
subplot(4,3,3), plot(f, angle(X_af), 'r')
ylabel('Radian')
x_b = pls_fn(2*t/tau + 1/2) - pls_fn(2*t/tau - 1/2);
X_bf = j*tau^2/2*pi*f.*sinc(tau/2*f).^2;
subplot(4,3,4), plot(t, x_b, 'm')
xlabel('t, s'), text(.5, .7, 'Odd')
subplot(4,3,5), plot(f, abs(X_bf), 'm')
xlabel('f, Hz'), title('|2\pi f sinc^2f|')
ylabel('Amplitude')
subplot(4,3,6), plot(f, angle(X_bf), 'm')
xlabel('f, Hz'), ylabel('Radian')
```


The result is



The square pulse (digital pulse send in a communication channel) and its Fourier transform the *sinc* function play important roles in signal and system analysis.

Energy Spectral Density In Chapter 1 the normalized energy of a signal was defined as

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} x^*(t)x(t) dt$$

Writing $x(t)$ in terms of its Fourier signal

$$\begin{aligned} E &= \int_{-\infty}^{\infty} x^*(t) \left[\int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df \right] dt \\ &= \int_{-\infty}^{\infty} X(f) \left[\int_{-\infty}^{\infty} x(t) e^{j2\pi ft} dt \right]^* df \\ &= \int_{-\infty}^{\infty} X(f) X^*(f) df = \int_{-\infty}^{\infty} |X(f)|^2 df \end{aligned}$$

Assuming $x(t)$ is a voltage signal, on a per ohm basis $|X(f)|^2$ has the unit of W-s/Hz or J/Hz, i.e., energy density with frequency, and we define

$$G(f) = |X(f)|^2$$

as *energy spectral density* which describes the distribution of signal energy over the frequency spectrum. The above integral over all frequency represent the total energy contained in the signal.

Example 4.2 (Example 4-2 textbook)

(a) Find the Fourier transform of $x(t) = e^{-\alpha t}u(t)$

(b) Using the energy spectral density find the total energy of the signal. which should be $E = \frac{1}{2\alpha}$, as found in Example 1-11(Text).

(c) Use MATLAB Symbolic Math feature to obtain the analytical expression for $X(f)$, and $G(f)$ for $\alpha = 2$.

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft} dt = \int_0^{\infty} e^{-\alpha t} e^{-j2\pi ft} dt \\ &= -\frac{1}{\alpha + j2\pi f} e^{-(\alpha + j2\pi f)t} \Big|_0^{\infty} = \frac{1}{\alpha + j2\pi f} \end{aligned}$$

(b) The energy spectral density is

$$G(f) = |X(f)|^2 = \frac{1}{\alpha^2 + (2\pi f)^2}$$

Energy contained in the signal within $-B < f < B$ is

$$\begin{aligned} E_B &= \int_{-B}^B \frac{1}{\alpha^2 + (2\pi f)^2} df \quad \text{let } v = \frac{2\pi f}{\alpha} \\ &= \frac{1}{\pi\alpha} \int_0^{\frac{2\pi B}{\alpha}} \frac{dv}{1 + v^2} = \frac{1}{\pi\alpha} \tan^{-1} \frac{2\pi B}{\alpha} \end{aligned}$$

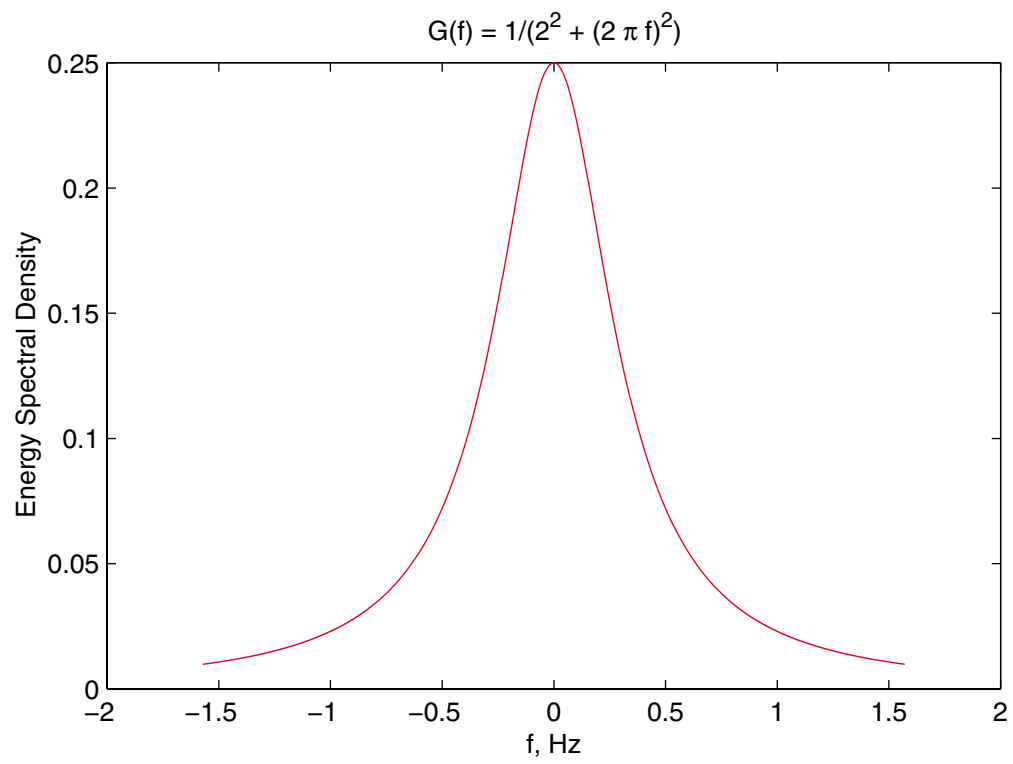
As $B \rightarrow \infty$, $E = \frac{1}{2\alpha}$, same as in Example 1-11(Text).

We use the following commands

```
% Example 4.2
syms t w f                                % Define symbolic variables
x = sym('exp(-2*t)*Heaviside(t)')         % Create the signal
X = fourier(x)                             % Fourier transform
Xf=subs(X,'w',sym('2*pi*f'))              %Change w r/s to f=w/(2*pi)Hz
Xf_conj=subs(Xf,'f',sym('-f'))            % Conj. Fourier transform
Gf = symmul(Xf, Xf_conj)                   % Energy Spectral Density
ezplot(Gf)                                 % Use ezplot to plot energy spectrum
```

The result is

```
x = exp(-2*t)*Heaviside(t)
X =
      1/(2+i*w)
Xf =
      1/(2+2*i*pi*f)
Xf_conj =
      1/(2-2*i*pi*f)
Gf =
      1/(2+2*i*pi*f)/(2-2*i*pi*f)
```



Condition for the existence of Fourier transform

The sufficient condition is similar to those given for Fourier series.

1. $x(t)$ must have a finite number of maxima and minima
2. $x(t)$ must have a finite number of discontinuity
3. $x(t)$ must be absolutely integrable

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty$$

From the last condition, we may say that any signal that meets the condition

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$$

is absolutely integrable. As you know any signal that has finite energy is known as *energy signal*. Energy signals generally include nonperiodic signals that have a finite time duration, or any signal that approaches zero as t approaches infinity, such as $x(t) = e^{-\alpha t}$.

Many signals in electrical engineering are not energy signals, such as step signal and all periodic signals and do not meet the Dirichlet condition. It can be shown that signals that have infinite energy, but contain a finite power, i.e., power signals do have a valid Fourier transform. We have seen that the average power of a power signal is

$$P = \lim_{t \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |x(t)|^2 dt < \infty$$

The average power is often used to describe the strength of the communication signals. A general characteristic of the power signal is that their Fourier transform contain impulse functions.

Fourier transform of an impulse

Fourier transform can be extended to apply to an impulse function. Consider the delayed impulse signal of strength A given by

$$x(t) = A\delta(t - t_0)$$

The Fourier transform is

$$\begin{aligned} X(f) &= \mathcal{F}\{A\delta(t - t_0)\} = \int_{-\infty}^{\infty} A\delta(t - t_0)e^{-j2\pi ft} dt \\ &= Ae^{-j2\pi ft_0} \end{aligned}$$

where in the above integral we made use of the sifting property of an impulse. Thus,

$$X(f) = \mathcal{F}\{A(\delta(t - t_0))\} = Ae^{-j2\pi ft_0}$$

For unit impulse function occurring at origin, $t_0 = 0$

$$X(f) = \mathcal{F}\{\delta(t)\} = 1$$

or in mathematical notation, we write

$$\delta(t) \xleftrightarrow{\mathcal{F}} 1$$

Now let us consider an impulse function in the frequency domain

$$X(f) = \delta(f - f_0)$$

The inverse Fourier transform of this function is given by

$$\begin{aligned} x(t) &= \mathcal{F}^{-1}\{\delta(f - f_0)\} = \int_{-\infty}^{\infty} \delta(f - f_0) e^{j2\pi ft} df \\ &= e^{j2\pi f_0 t} \end{aligned}$$

Thus, we have

$$x(t) = \mathcal{F}^{-1}\{\delta(f - f_0)\} = e^{j2\pi f_0 t}$$

or

$$e^{j2\pi f_0 t} \xleftrightarrow{\mathcal{F}} \delta(f - f_0)$$

Fourier Transform Theorems (Properties of the Fourier transform)

Fourier transform has several properties that are useful for obtaining additional Fourier transform.

Linearity (superposition) Theorem

Because Fourier transform and its inverse involve integral operation, principle of superposition applies, i.e., given

$$x_1(t) \xleftrightarrow{\mathcal{F}} X_1(f) \quad \text{and} \quad x_2(t) \xleftrightarrow{\mathcal{F}} X_2(f)$$

then

$$[a_1x_1(t) + a_2x_2(t)] \xleftrightarrow{\mathcal{F}} [a_1X_1(f) + a_2X_2(f)]$$

Example 4.3

Using the Linear property find the Fourier transform of $x(t) = A \cos 2\pi f_0 t$.

Writing $x(t)$ in terms of the counter rotating phasors

$$A \cos 2\pi f_0 t = \frac{A}{2} [e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}]$$

we have

$$\begin{aligned} \mathcal{F}\{A \cos 2\pi f_0 t\} &= \frac{A}{2} [\mathcal{F}\{e^{j2\pi f_0 t}\} + \mathcal{F}\{e^{-j2\pi f_0 t}\}] \\ &= \frac{A}{2} [\delta(f - f_0) + \delta(f + f_0)] \end{aligned}$$

or

$$A \cos 2\pi f_0 t \xleftrightarrow{\mathcal{F}} \frac{A}{2} [\delta(f - f_0) + \delta(f + f_0)]$$

Time-Delay Theorem

Given

$$x(t) \xleftrightarrow{\mathcal{F}} X(f)$$

then

$$x(t - t_0) \xleftrightarrow{\mathcal{F}} X(f)e^{-j2\pi ft_0}$$

Example 4.4(Example 4-4 Textbook)

Obtain the Fourier transform of a unit-high square pulse 2 unit wide starting at $t = 0$.

This signal can be obtained from $x_a(t)$ of Example 4.1 by delaying it by $t_0 = \frac{\tau}{2} = 1$ unit of time. Applying the time-delay theorem to the Fourier transform of $x_a(t)$, we get

$$\begin{aligned} X(f) &= \tau \operatorname{sinc}(\tau f) e^{-j2\pi ft_0} \\ &= 2 \operatorname{sinc}(2f) e^{-j2\pi f} \end{aligned}$$

Scale Changing Theorem

Given $x(t) \xleftrightarrow{\mathcal{F}} X(f)$

then, $x(at) \xleftrightarrow{\mathcal{F}} \frac{1}{|a|} X\left(\frac{f}{a}\right)$

To prove this, from the defining integral

$$\mathcal{F}\{x(at)\} = \int_{-\infty}^{\infty} x(at)e^{-j2\pi ft} dt$$

For $a > 0$, substituting $\tau = at$, then $d\tau = a dt$, we have

$$\mathcal{F}\{x(\tau)\} = \frac{1}{a} \int_{-\infty}^{\infty} x(\tau)e^{-j2\pi \frac{f}{a}\tau} d\tau$$

or we can write

$$\mathcal{F}\{x(at)\} = \frac{1}{a} X\left(\frac{f}{a}\right)$$

Replacing $\frac{1}{a}$ by $\frac{1}{|a|}$, makes this relation applicable for both positive and negative value of a .

$$\mathcal{F}\{x(at)\} = \frac{1}{|a|} X\left(\frac{f}{a}\right)$$

Compression in one domain leads to a stretching and an amplitude reduction in the other.

Example 4.5

Consider the FT pair

$$x(t) = e^{-t}u(t) \xleftrightarrow{\mathcal{F}} \frac{1}{1 + j\omega}$$

Scaling $x(t)$ to $x(\alpha t)$ for $\alpha > 0$, gives

$$x(\alpha t) = e^{-\alpha t}u(t) \xleftrightarrow{\mathcal{F}} \frac{1}{\alpha} \frac{1}{1 + j\frac{\omega}{\alpha}} = \frac{1}{\alpha + j\omega}$$

Example 4.6

In Example 4.1, the FT of the signal $x_a(t)$ was found

$$x_a(t) = \Pi\left(\frac{t}{\tau}\right) \xleftrightarrow{\mathcal{F}} \tau \operatorname{sinc}(\tau f)$$

Scaling for $a = \frac{\tau}{2}$, we get

$$x_a(at) = \Pi\left(\frac{t}{2}\right) \xleftrightarrow{\mathcal{F}} 2 \operatorname{sinc}(2f)$$

Duality Theorem

The FT and IFT were defined as

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft} dt \quad x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft} df$$

interchanging t with f , results in

$$X(t) = \int_{-\infty}^{\infty} x(f)e^{-j2\pi ft} df \quad x(f) = \int_{-\infty}^{\infty} X(t)e^{j2\pi ft} dt$$

In order to maintain the same form in the kernel, we replace t by $-t$ in the first relation and f by $-f$ in the second relation, we obtain

$$X(-t) = \int_{-\infty}^{\infty} x(f)e^{j2\pi ft} df \quad x(-f) = \int_{-\infty}^{\infty} X(t)e^{-j2\pi ft} dt$$

From the above we see that

$$\text{if } x(t) \xleftrightarrow{\mathcal{F}} X(f), \quad \text{then } X(t) \xleftrightarrow{\mathcal{F}} x(-f)$$

For even function, we use $t \xleftrightarrow{\mathcal{F}} f$.

Example 4.7(Example 4-7 Textbook)

Given the transform pair

$$x(t) = \Pi\left(\frac{t}{\tau}\right) \xleftrightarrow{\mathcal{F}} \tau \operatorname{sinc}(\tau f) = X(f)$$

We replace f in $X(f)$ by t to get the new time function

$$X(t) = \tau \operatorname{sinc}(\tau t)$$

Then, from the duality theorem, the FT of this new time function is

$$x(f) = \Pi\left(\frac{t}{\tau}\right) \Big|_{t \rightarrow -f} = \Pi\left(\frac{-f}{\tau}\right)$$

Let $W = \frac{\tau}{2}$, and since the signal is even $\Pi\left(\frac{-f}{\tau}\right) = \Pi\left(\frac{f}{\tau}\right)$, we get

$$2W \operatorname{sinc}(2Wt) \xleftrightarrow{\mathcal{F}} \Pi\left(\frac{f}{2W}\right)$$

Frequency Translation Theorem

The frequency translation theorem is stated mathematically as

$$x(t)e^{j2\pi f_0 t} \xleftrightarrow{\mathcal{F}} X(f - f_0)$$

This means that multiplication by exponential in time domain corresponds to translation in frequency domain.

Example 4.8 (Example 4-8 Textbook)

(a) Find the FT of

$$x_1(t) = \Pi\left(\frac{t}{2}\right) e^{j20\pi t}$$

We know that

$$\mathcal{F}\left\{\Pi\left(\frac{t}{2}\right)\right\} = 2\text{sinc}(2f)$$

From frequency translation theorem, we have

$$X_1(f) = 2\text{sinc}[2(f - 10)]$$

(b) Find the FT of

$$x_2(t) = \Pi\left(\frac{t}{2}\right) \cos 20\pi t$$

We can write $x_2(t)$ as

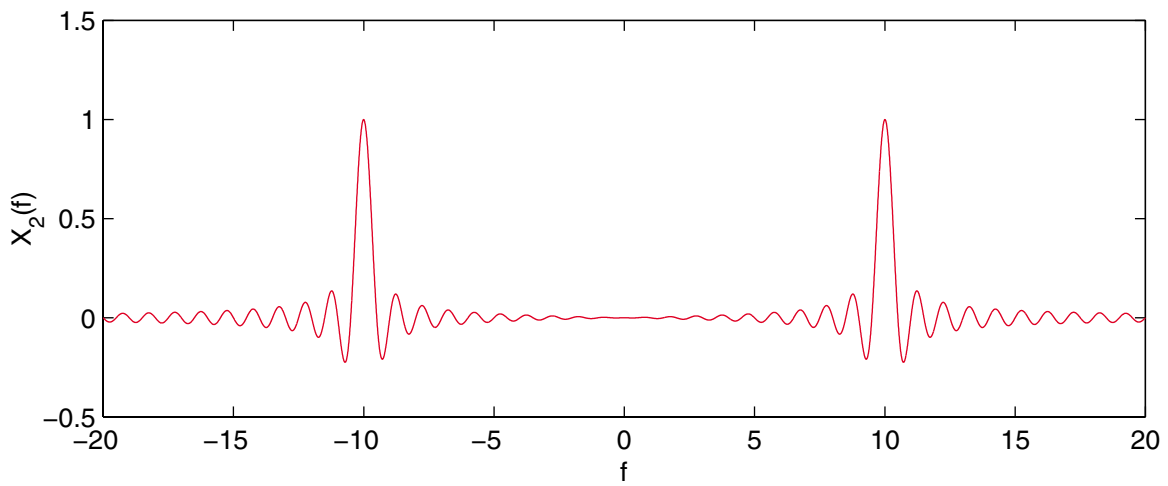
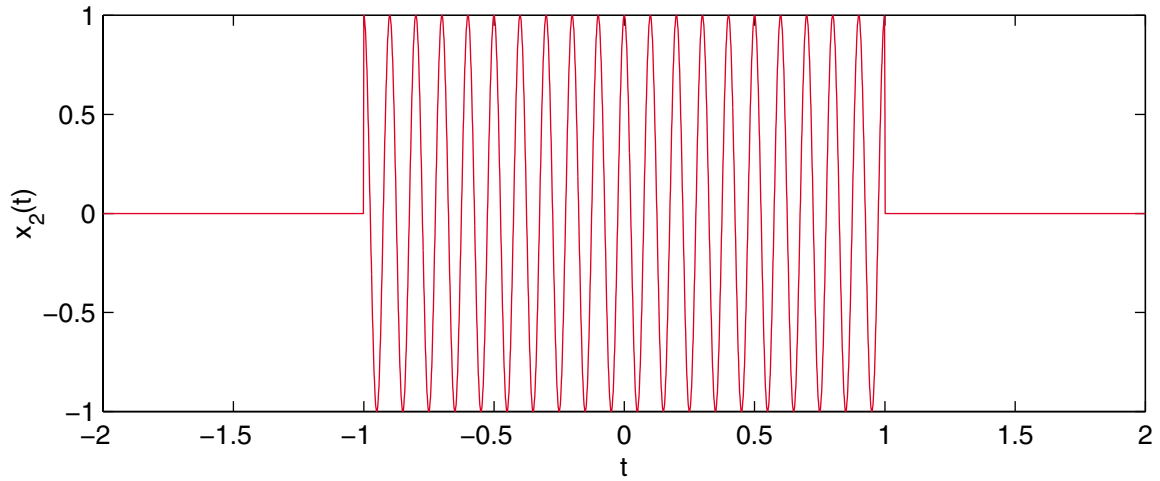
$$\begin{aligned} x_2(t) &= \frac{1}{2} \Pi\left(\frac{t}{2}\right) \frac{1}{2} (e^{j20\pi t} + e^{-j20\pi t}) \\ &= \frac{1}{2} \Pi\left(\frac{t}{2}\right) e^{j20\pi t} + \frac{1}{2} \Pi\left(\frac{t}{2}\right) e^{-j20\pi t} \end{aligned}$$

From frequency translation theorem, we have

$$X_2(f) = \text{sinc}[2(f - 10)] + \text{sinc}[2(f + 10)]$$

We use the following commands to plot $x_2(t)$, and $X_2(f)$

```
%chs4_ex8.m
t=-2:.001:2;
f=-20:.005:20;
x2 = pls_fn(t/2).*cos(20*pi*t);
X2 = sinc(2*(f-10))+ sinc(2*(f+10));
subplot(2,1,1), plot(t,x2), xlabel('t'), ylabel('x_2(t)')
subplot(2,1,2), plot(f,X2), xlabel('f'), ylabel('X_2(f)')
```



Example 4.9

In a communication system two signals

$$x_1(t) = 4 \cos(200\pi t) \text{ and } x_2(t) = 6 \cos(800\pi t)$$

are multiplied to produce

$$x(t) = 24 \cos(200\pi t) \cos(800\pi t)$$

Find the FT and frequency spectrum of $x(t)$.

Using Euler's identity, we rewrite $x(t)$ as

$$\begin{aligned} x(t) &= 24 \cos(200\pi t) \frac{1}{2} (e^{j800\pi t} + e^{-j800\pi t}) \\ &= 12 \cos(200\pi t) e^{j800\pi t} + 12 \cos(200\pi t) e^{-j800\pi t} \end{aligned}$$

From the frequency translation theorem, FT of $x(t)$ is

$$\begin{aligned} X(f) &= 6 [\delta(f - 100 - 400) + \delta(f + 100 - 400)] \\ &\quad + 6 [\delta(f - 100 + 400) + \delta(f + 100 + 400)] \end{aligned}$$

or

$$X(f) = 6 [\delta(f - 500) + \delta(f - 300) \\ + \delta(f + 300) + \delta(f + 500)]$$

It is interesting to observe that the inverse FT of $X(f)$ is

$$x(t) = \mathcal{F}\{6 [\delta(f - 500) + \delta(f + 500)]\} \\ + \mathcal{F}\{6 [\delta(f - 300) + \delta(f + 300)]\} \\ = 12 \cos 1000\pi t + 12 \cos 600\pi t$$

Form the above result, we can see that the product of two cosine signals has produced the sum of two cosine signals, one with the sum of the two frequencies and the other their difference. This can be verified by the use of trigonometric identity. This characteristic is often used in radar and sonar applications.

The **fourier** function of the Symbolic Math toolbox is used to obtain the FT of $x(t)$ in Example 4.9. Also, **ifourier** function is used to find the signal from its FT, and the result is simplified.

We use the following statements

```
% chs4_ex9.m
syms t w          % Define symbolic variables
x1= sym('4*cos(200*pi*t)');
x2= sym('6*cos(800*pi*t)');
x = x1*x2         % Product of the signals
X = fourier(x)    % Fourier transform
g= ifourier(X);   % Inverse Fourier transform
x= simplify(g)    % Simplify the expression
```

The result is

```
x = 24*cos(200*pi*t)*cos(800*pi*t)
X = 12*pi*Dirac(w-1000*pi)+12*pi*Dirac(w+600*pi)+
    12*pi*Dirac(w-600*pi)+12*pi*Dirac(w+1000*pi)
x =
    12*cos(600*pi*t)+12*cos(1000*pi*t)
```

Writing in terms of frequency f , the result is

$$X(f) = 6\delta(f-500) + 6\delta(f+300) + 6\delta(f-300) + 6\delta(f+500)$$

Fourier Transform of Periodic Signals

In Chapter 3, we showed that a periodic signal can be represented by its Fourier series

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn2\pi f_0 t} \quad \text{where} \quad X_n = \frac{1}{T_0} \int_{T_0} x(t) e^{-jn2\pi f_0 t} dt$$

From Equation (1) the Fourier transform of $x(t)$ is

$$X(f) = \int_{-\infty}^{\infty} \left[\sum_{n=-\infty}^{\infty} X_n e^{j2\pi n f_0 t} \right] e^{-j2\pi f t} dt = \sum_{n=-\infty}^{\infty} X_n \int_{-\infty}^{\infty} e^{j2\pi n f_0 t} e^{-j2\pi f t} dt$$

Previously we found the Fourier transform pair

$$e^{j2\pi f_0 t} \xleftrightarrow{\mathcal{F}} \delta(f - f_0)$$

Using this relation, we get

$$\sum_{n=-\infty}^{\infty} X_n e^{j2\pi n f_0 t} \xleftrightarrow{\mathcal{F}} \sum_{n=-\infty}^{\infty} X_n \delta(f - n f_0)$$

This is an important result, showing that the frequency spectrum of a periodic signal is a series of impulses of weight X_n located at the corresponding harmonic frequency $n f_0$.

Example 4.10 (Example 4-10)

Find the Fourier transform of a periodic train of impulses referred to as *ideal sampling waveform*

$$y_s(t) = \sum_{m=-\infty}^{\infty} \delta(t - mT_s)$$

The complex Fourier coefficients are given by

$$Y_n = \frac{1}{T_s} \int_{-\frac{T_s}{2}}^{\frac{T_s}{2}} \delta(t) e^{-j2\pi n f_s t} dt = \frac{1}{T_s} = f_s$$

From the result of the Fourier transform pair of the periodic signal, we get

$$\sum_{m=-\infty}^{\infty} \delta(t - mT_s) \xleftrightarrow{\mathcal{F}} f_s \sum_{n=-\infty}^{\infty} \delta(f - n f_0)$$

Differentiation Theorem

$$\begin{aligned} \text{Given} \quad & x(t) \xleftrightarrow{\mathcal{F}} X(f) \\ \text{then} \quad & \frac{d[x(t)]}{dt} \xleftrightarrow{\mathcal{F}} j2\pi f X(f) \\ \text{also} \quad & \frac{d^n[x(t)]}{dt^n} \xleftrightarrow{\mathcal{F}} (j2\pi f)^n X(f) \end{aligned}$$

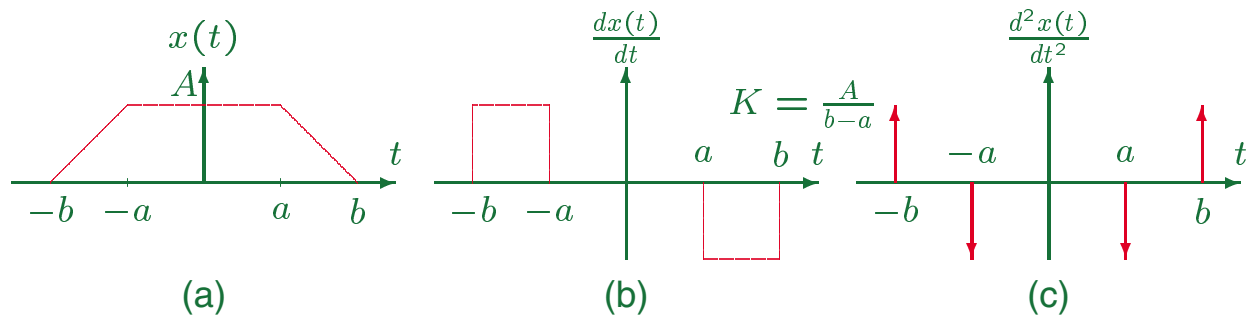
This theorem can be proven easily by differentiating both sides of Equation (1).

Integration Theorem

$$\begin{aligned} \text{Given} \quad & x(t) \xleftrightarrow{\mathcal{F}} X(f) \\ \text{then} \quad & \int_{-\infty}^{\infty} x(\tau) d\tau \xleftrightarrow{\mathcal{F}} \frac{1}{j2\pi f} X(f) + \frac{X(0)\delta(f)}{2} \\ \text{where} \quad & X(0) = \int_{-\infty}^{\infty} x(t) dt \end{aligned}$$

An important observation is: due to the $\frac{1}{f}$ factor, integrating a signal decreases its high frequency content.

Example 4.11 (Example 4-11 text) Find the FT of the trapezoidal signal shown in Figure (a)



The first and second derivative of this signal is shown in Figures (b)&(c). The analytical expression for the second derivative is

$$\frac{d^2x(t)}{dt^2} = K[\delta(t + b) - \delta(t + a) - \delta(t - a) + \delta(t - b)]$$

The FT of the second derivative is

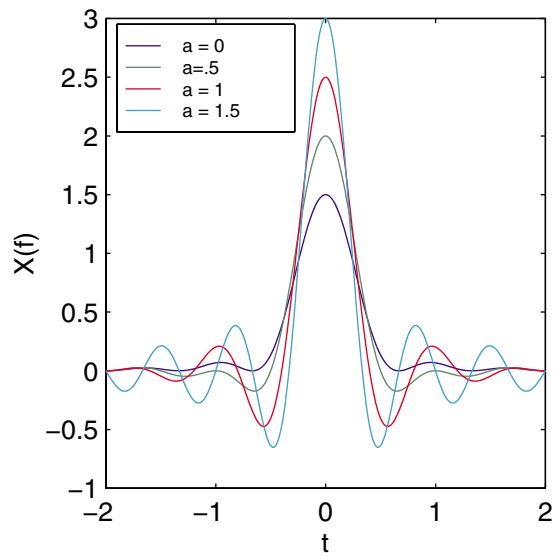
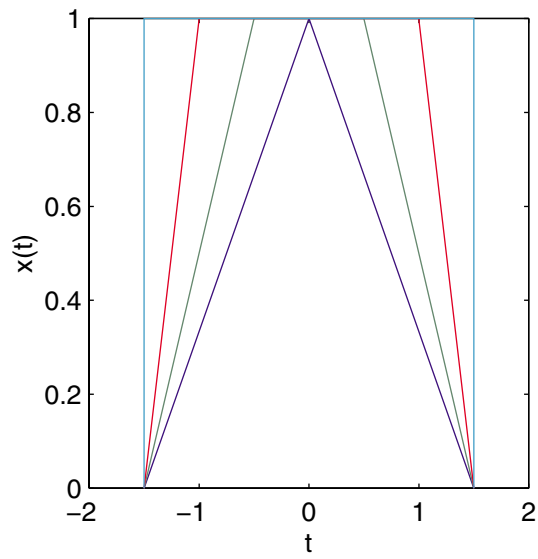
$$\begin{aligned} \mathcal{F}\left[\frac{d^2x(t)}{dt^2}\right] &= K[e^{j2\pi fb} - e^{j2\pi fa} - e^{-j2\pi fa} + e^{-j2\pi fb}] \\ &= 2K[\cos 2\pi fb - \cos 2\pi fa] = (j2\pi f)^2 X(f) \end{aligned}$$

Therefore,

$$\begin{aligned} X(f) &= 2K \left[\frac{\cos 2\pi fb - \cos 2\pi fa}{(j2\pi f)^2} \right] \\ &= K \left[b^2 \frac{\sin^2 \pi fb}{(\pi fb)^2} - a^2 \frac{\sin^2 \pi fa}{(\pi fa)^2} \right] \\ &= K [b^2 \text{sinc}^2(fb) - a^2 \text{sinc}^2(fa)] \end{aligned}$$

The following MATLAB commands obtains the plot of $x(t)$ and $X(f)$ for $A = 1$, $b = 1.5$, and four values of $a : 0, 0.5, 1, 1.5$

```
%chs4_ex11.m
f=-2:.001:2;
A =1;
b=1.5;
a= -0.500000001;
for m = 1:4
a = a + 0.5;
t(m,:) = [-b  -a  0  a  b];
x(m,:) = [ 0  A  A  A  0];
K = A/(b-a);
X(m,:) = K*(b^2*sinc(f*b).*sinc(f*b)- a^2*sinc(f*a).*sinc(f*a))
end
subplot(1,2,1), plot(t(1,:), x(1,:), t(2,:), x(2,:), t(3,:),
xlabel('t'), ylabel('x(t)')
subplot(1,2,2), plot(f, X), xlabel('t'), ylabel('X(f)')
legend('a = 0', 'a=.5', 'a = 1', 'a = 1.5',2)
```



Example 4.12 (Example 4-12 Textbook)

Using the integration theorem, find the FT of a unit step function.

$$\mathcal{F}[\delta(t)] = 1$$

From the integration theorem, we have

$$\mathcal{F}\left[\int_{-\infty}^{\infty} x(\tau)d\tau\right] = \frac{1}{j2\pi f}X(f) + \frac{X(0)\delta(f)}{2}$$

since

$$u(t) = \int_{-\infty}^t \delta(\lambda)d\lambda$$

and

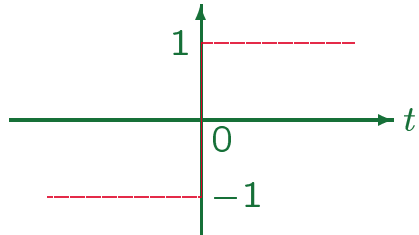
$$X(0) = \int_{-\infty}^{\infty} \delta(t)dt = 1$$

then

$$\mathcal{F}[u(t)] = \frac{1}{j2\pi f} + \frac{1}{2}\delta(f)$$

Fourier transform of the Signum function

This is a signal related to the unit step signal, and is defined as

$$\text{sgn}(t) = 2u(t) - 1 = \begin{cases} 1 & t > 0 \\ -1 & t < 0 \end{cases}$$


$$\frac{d}{dt} \text{sgn}(t) = 2\delta(t)$$

From the differentiation theorem, we have

$$\mathcal{F} \left[\frac{dx(t)}{dt} \right] = j2\pi f X(f)$$

Since $\mathcal{F} \left[\frac{d}{dt} \text{sgn}(t) \right] = \mathcal{F}[2\delta(t)] = 2$
we have $j2\pi f X(f) = 2$
or $\mathcal{F}[\text{sgn}(t)] = X(f) = \frac{1}{j\pi f}$

and from the duality theorem, we can write

$$\mathcal{F} \left[\frac{1}{\pi t} \right] = -j \text{sgn}(f)$$

Fourier transform of a constant (DC level)

We found that

$$\mathcal{F} \left[e^{j2\pi f_0 t} \right] = \delta(f - f_0)$$

For $f_0 = 0$, we get

$$\mathcal{F}[1] = \delta(f) \quad \text{or} \quad \mathcal{F}[K] = K\delta(f)$$

Knowing the FT of a signum function and unity, we can obtain the FT of a step function. We found the FT of a signum function as

$$\mathcal{F}[2u(t) - 1] = \frac{1}{j\pi f}$$

or

$$2\mathcal{F}[u(t)] - \mathcal{F}[1] = \frac{1}{j\pi f}$$

thus

$$\begin{aligned} \mathcal{F}[u(t)] &= \frac{1}{2} \frac{1}{j\pi f} + \frac{1}{2} \mathcal{F}[1] \\ &= \frac{1}{j2\pi f} + \frac{1}{2} \delta(f) \end{aligned}$$

This is the same result as was found in Example 4.12

Convolution Theorem

This is given by

$$\mathcal{F}[x_1(t) * x_2(t)] = X_1(f)X_2(f)$$

$$\begin{aligned}\mathcal{F}[x_1(t) * x_2(t)] &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x_1(t - \lambda) u(t - \lambda) x_2(\lambda) d\lambda \right] e^{-j2\pi f t} dt \\ &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x_1(t - \lambda) e^{-j2\pi f(t - \lambda)} dt \right] x_2(\lambda) e^{-j2\pi f \lambda} d\lambda \\ &= X_1(f)X_2(f)\end{aligned}$$

We write

$$x_1(t) * x_2(t) \xleftrightarrow{\mathcal{F}} X_1(f)X_2(f)$$

Multiplication Theorem

According to the duality theorem, we can also write

$$x_1(t)x_2(t) \xleftrightarrow{\mathcal{F}} X_1(f) * X_2(f)$$

Example 4.13 (Example 4-13 Textbook)

Consider the FT of the convolution of two rectangular pulses which is a triangular signal given by

$$\Pi\left(\frac{t}{\tau}\right) * \Pi\left(\frac{t}{\tau}\right) = \tau \Lambda\left(\frac{t}{\tau}\right)$$

$$\text{Since } \mathcal{F}\left[\Pi\left(\frac{t}{\tau}\right)\right] = \tau \text{sinc}(f\tau)$$

Then

$$\mathcal{F}\left[\tau \Lambda\left(\frac{t}{\tau}\right)\right] = \left\{\mathcal{F}\left[\Pi\left(\frac{t}{\tau}\right)\right]\right\}^2 = \tau^2 \text{sinc}^2(f\tau)$$

Example 4.14 (Example 4-14 Textbook)

Find the FT of the convolution of the following signal, known as the Hilbert transform

$$\hat{x}(t) = x(t) * \frac{1}{\pi t} = \frac{1}{\pi} \int_{-\infty}^{\infty} x(\lambda) \frac{1}{t - \lambda} d\lambda$$

Since $\mathcal{F}\left[\frac{1}{\pi t}\right] = -j \operatorname{sgn}(f)$, using the convolution theorem, we get

$$\mathcal{F}[\hat{x}(t)] = \mathcal{F}\left[\frac{1}{\pi t}\right] \mathcal{F}[x(t)] = -j \operatorname{sgn}(f) X(f)$$

Example 4.15 (Example 4-15)

Find the FT of the cosinusoidal pulse, given by

$$x(t) = A\Pi\left(\frac{t}{\tau}\right) \cos 2\pi f_0 t$$

We have the following FT

$$A\Pi\left(\frac{t}{\tau}\right) \xleftrightarrow{\mathcal{F}} A\tau \text{sinc}(f\tau)$$

and

$$\cos 2\pi f_0 t \xleftrightarrow{\mathcal{F}} \frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$$

From the multiplication theorem

$$\mathcal{F}[x(t)] = A\tau \text{sinc}(f\tau) * \left[\frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)\right]$$

The result for the following convolution is

$$\begin{aligned} \text{sinc}(f\tau) * \delta(f \pm f_0) &= \int_{-\infty}^{\infty} \text{sinc}(\lambda\tau) \delta(\lambda - f \pm f_0) d\lambda \\ &= \text{sinc}(f \pm f_0)\tau \end{aligned}$$

Thus,

$$\mathcal{F}[x(t)] = \frac{A\tau}{2} [\text{sinc}(f - f_0)\tau + \text{sinc}(f + f_0)\tau]$$

Frequency Differentiation

The differentiation theorem has a dual property and we can write

$$\frac{dX(f)}{df} \xleftrightarrow{\mathcal{F}} -j2\pi tx(t)$$

Example 4.16

Find the FT of

$$g(x) = te^{-\alpha t}u(t)$$

$$X(f) = \mathcal{F} [e^{-\alpha t}u(t)] = \frac{1}{\alpha + j2\pi f}$$

Therefore

$$\begin{aligned} G(f) &= \mathcal{F}[tx(t)] = \frac{1}{-j2\pi f} \left[\frac{d}{df} X(f) \right] \\ &= \frac{1}{-j2\pi} \left[\frac{d}{df} \left(\frac{1}{\alpha + j2\pi f} \right) \right] = \frac{1}{-j2\pi} \left[\frac{-j2\pi}{(\alpha + j2\pi f)^2} \right] \\ &= \frac{1}{(\alpha + j2\pi f)^2} \end{aligned}$$

Switched Cosine

This is a signal turned on at $t = 0$, $x(t) = \cos(2\pi f_0 t)u(t)$. We can write this as

$$x(t) = \frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2}u(t) = \frac{1}{2}e^{j2\pi f_0 t}u(t) + \frac{1}{2}e^{-j2\pi f_0 t}u(t)$$

The Fourier transform of $u(t)$ is

$$\mathcal{F}[u(t)] = \frac{1}{j2\pi f} + \frac{1}{2}\delta(f)$$

Then from the frequency shifting property, we get

$$\begin{aligned}\mathcal{F}[x(t)] &= \frac{1}{2} \left[\frac{1}{j2\pi(f - f_0)} + \frac{1}{2}\delta(f - f_0) \right] \\ &+ \frac{1}{2} \left[\frac{1}{j2\pi(f + f_0)} + \frac{1}{2}\delta(f + f_0) \right] \\ &= \frac{1}{4}[\delta(f - f_0) + \delta(f + f_0)] + \frac{1}{j2\pi f} \left(\frac{f}{f^2 - f_0^2} \right)\end{aligned}$$

Modulation

Transmission of a low frequency message signal requires shifting the range of frequencies contained in the message signal to a much higher frequencies. This is achieved by modulation and transmitting the modulated signal. Demodulation is used for recovering the modulated signal at the receiving end. Among the many schemes available are amplitude modulation (AM), frequency modulation (FM), and phase modulation (PM).

Amplitude modulation is the process of multiplying a low-frequency signal $x(t)$, such as bandlimited voice or music signals by a high frequency sinusoid known as *carrier signal*.

$$x_{AM}(t) = x(t) \cos 2\pi f_0 t$$

Suppose we want to transmit a speech signal $x(t)$ which has a spectrum with frequencies ranging from 200 Hz to 4 kHz over a communication channel consisting of a satellite link. One method is to multiply $x(t)$ with a carrier frequency in the range of frequency used by the satellite (few hundred MHz to 40 GHz).

The modulated Signal follows the amplitude of $x(t)$. In the frequency domain, the frequency spectrum of the signal moves to a new center frequency. This will allow us to modulate many signals with different carrier frequencies and combine them for transmission over a communication channel. This process is known as frequency multiplexing.

Example 4.17

Suppose we want to transmit a signal

$$x(t) = 4 + 4 \cos 2\pi 20t$$

over a communication channel operating at 400 Hz.

The modulated signal is

$$\begin{aligned}x_{AM}(t) &= (4 + 4 \cos 40\pi t) \cos 800\pi t \\ &= 4 \cos 800\pi t + 4 \cos 40\pi t \cos 800\pi t \\ &= 4 \cos 800\pi t + 2 \cos 840\pi t + 2 \cos 760\pi t\end{aligned}$$

$x_{AM}(t)$ in term of the counter rotating phasor is

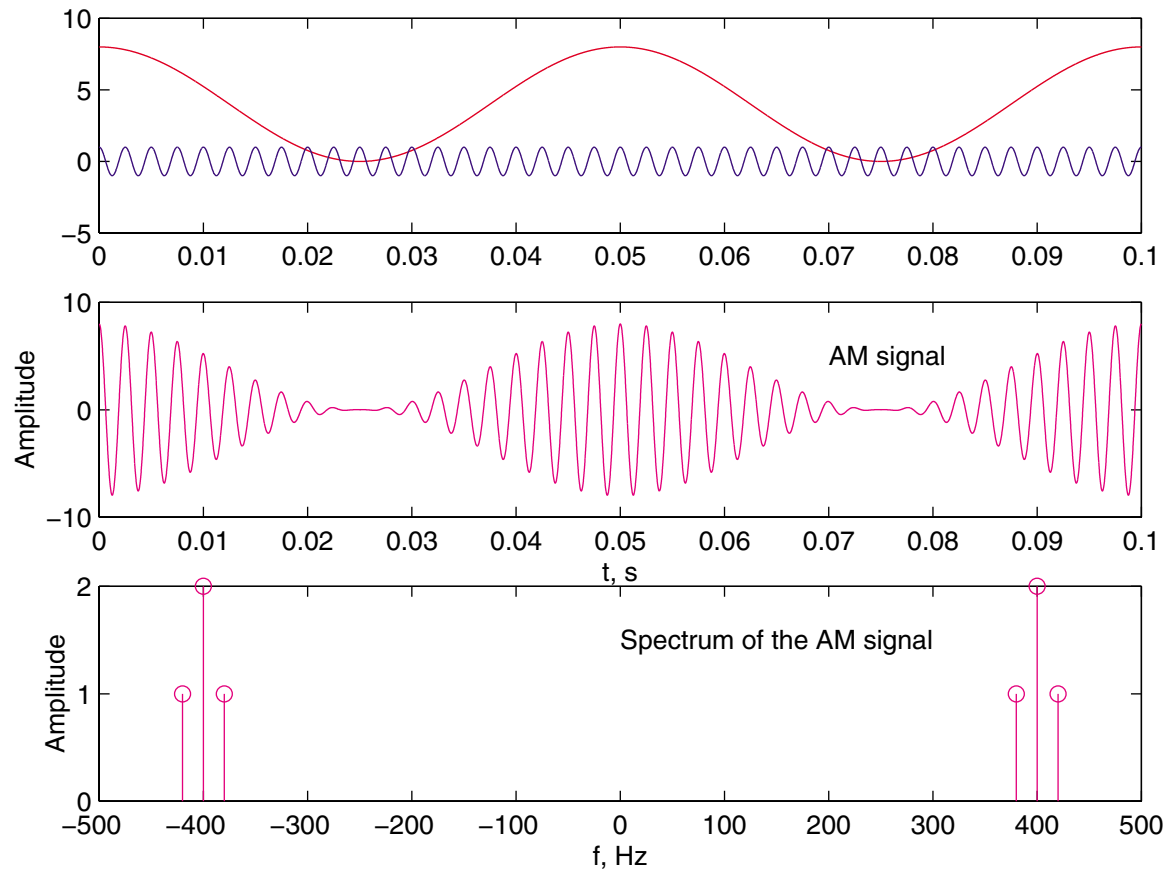
$$\begin{aligned}x(t) &= \left[2e^{j800\pi t} + 2e^{-j800\pi t} \right] \\ &= + \left[e^{j840\pi t} + e^{-j840\pi t} \right] + \left[e^{j760\pi t} + e^{-j760\pi t} \right]\end{aligned}$$

The following statements are used to plot $x(t)$, the carrier signal, the modulated signal $x_{AM}(t)$, and the amplitude spectrum of the AM signal.


```

%chs4_ex17.m
t=0:.00001:.1;
x =4+4*cos(2*pi*20*t);
cs= cos(2*pi*400*t);
x_AM = x.*cs;
subplot(3,1, 1), plot(t,x,'r', t, cs,'b')
subplot(3,1,2), plot(t, x_AM, 'm')
xlabel('t, s'), ylabel('Amplitude')
text(0.07, 5, 'AM signal')
f=[380 400 420]; f=[-f f];
A = [2, 4, 2]; A= [A/2, A/2];
subplot(3,1,3), stem(f, A, 'm')
xlabel('f, Hz'), ylabel('Amplitude')
text(0, 1.5, 'Spectrum of the AM signal')

```



The signal processing toolbox provides the following functions:

`[y, t] = modulate(x, fc, fs, 'method', opt)`

`x = demod(y, fc, fs, 'method', opt)`

where

fs is the sampling frequency.

fc is the carrier frequency.

method is the desired modulation method.

opt is the optional argument that the method requires.

Ten methods are available, few of these methods are

am Amplitude modulation

fm Frequency modulation

ptm Pulse time modulation

pwm Pulse width modulation

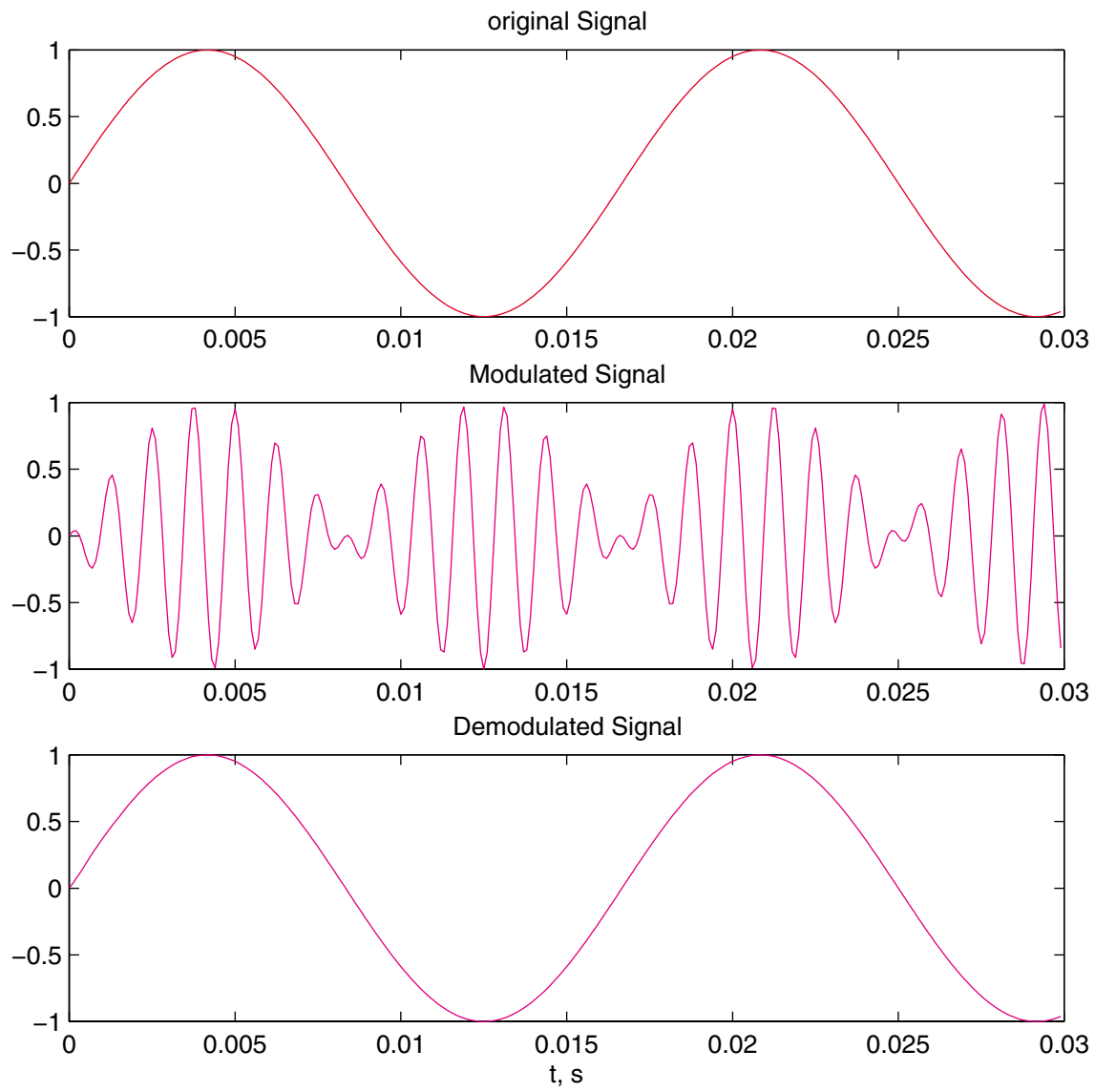
qam quadrature amplitude modulation

Example 4.18

A 60 Hz sin wave is sampled at 10000Hz. Obtain and plot the modulated and demodulated signals. The carrier signal frequency is 800 Hz.

We use the following commands and plot a portion of the signals:

```
%chs4_ex18.m
t=(0:1/10000:4);
x =sin(2*pi*60*t);
y = modulate(x, 800, 10000, 'am');
subplot(3,1, 1), plot(t(1:300) ,x(1:300),'r')
title('original Signal')
subplot(3,1,2), plot(t(1:300), y(1:300), 'm')
title('Modulated Signal')
xd = 2*demod(y, 800, 10000, 'am');
subplot(3,1,3), plot(t(1:300), xd(1:300), 'm')
axis([0 .03 -1 1])
title('Demodulated Signal')
xlabel('t, s')
```



Fourier Transform of a Modulated Signal

$$x_{AM}(t) = x(t) \cos 2\pi f_0 t$$

The Fourier transform of $\cos 2\pi f_0 t$ is

$$\mathcal{F}[\cos 2\pi f_0 t] = \frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$$

Multiplication in time domain corresponds to convolution in the frequency domain, and we have

$$\begin{aligned}\mathcal{F}[x_{AM}(t)] &= \mathcal{F}[x(t) \cos 2\pi f_0 t] \\ &= \mathcal{F}[x(t)] * \mathcal{F}[\cos 2\pi f_0 t] \\ &= X(f) * \left[\frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0) \right] \\ &= \frac{1}{2} \int_{-\infty}^{\infty} X(\lambda) \delta(f - f_0 - \lambda) d\lambda \\ &\quad + \frac{1}{2} \int_{-\infty}^{\infty} X(\lambda) \delta(f + f_0 - \lambda) d\lambda \\ &= \frac{1}{2} X(f - f_0) + \frac{1}{2} X(f + f_0)\end{aligned}$$

or

$$x(t) \cos 2\pi f_0 t \xleftrightarrow{\mathcal{F}} \frac{1}{2} X(f - f_0) + \frac{1}{2} X(f + f_0)$$

We see that the FT of the modulated signal is half the sum of the shifted transform $X(f - f_0)$ and $X(f + f_0)$

Fourier Transform Pairs

$x(t)$	$X(f)$
$\dot{\delta}(t)$	$j2\pi f$
$\delta(t)$	1
$\delta(t - t_0)$	$e^{-j2\pi f t_0}$
$e^{j2\pi f_0 t}$	$\delta(f - f_0)$
1	$\delta(f)$
$\Pi\left(\frac{t}{\tau}\right)$	$\tau \operatorname{sinc}(\tau f)$
$2W \operatorname{sinc}(2Wt)$	$\Pi\left(\frac{f}{2W}\right)$
$\Lambda\left(\frac{t}{\tau}\right)$	$\tau \operatorname{sinc}^2(\tau f)$
$\operatorname{sinc}(t)$	$\Pi(f)$
$u(t)$	$\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$
$\operatorname{sgn}(t)$	$\frac{1}{j\pi f}$
$\frac{1}{\pi t}$	$-j \operatorname{sgn}(f)$
$e^{-\alpha t} u(t)$	$\frac{1}{\alpha + j2\pi f}$
$t e^{-\alpha t} u(t)$	$\frac{1}{(\alpha + j2\pi f)^2}$
$e^{-\alpha t }$	$\frac{2\alpha}{\alpha^2 + (j2\pi f)^2}$
$e^{-\pi\left(\frac{t}{\tau}\right)^2}$	$e^{-\pi\left(\frac{f}{\tau}\right)^2}$
$\cos 2\pi f_0 t$	$\frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$
$\sin 2\pi f_0 t$	$\frac{1}{2j}\delta(f - f_0) - \frac{1}{2j}\delta(f + f_0)$
$tu(t)$	$\frac{j}{4\pi}\dot{\delta}(f) - \frac{1}{4\pi^2 f^2}$
$\hat{x}(t) = x(t) * \frac{1}{\pi t}$	$-j \operatorname{sgn}(f) X(f)$
$\sum_{m=-\infty}^{\infty} \delta(t - mT_s)$	$\frac{1}{T_s} \sum_{m=-\infty}^{\infty} \delta(f - mf_s)$

Fourier Transform Theorems (Operational Transforms)

Operational Properties of the Fourier Transform		
Transform	$x(t)$	$X(f)$
Duality	$X(t)$	$x(-f)$
Superposition	$a_1x_1(t) + a_2x_2(t)$	$a_1X_1(f) + a_2X_2(f)$
Time delay	$x(t - t_0)$	$X(f)e^{-j2\pi ft_0}$
Time reversal	$x(-t)$	$X(-f)$
Scaling	$x(at)$	$\frac{1}{ a }X\left(\frac{f}{a}\right)$
Frequency translation	$x(t)e^{j2\pi f_0t}$	$X(f - f_0)$
Modulation	$x(t)\cos(2\pi f_0t)$	$\frac{1}{2}X(f - f_0) + \frac{1}{2}X(f + f_0)$
Derivative	$\frac{dx(t)}{dt}$	$j2\pi fX(f)$
	$\frac{d^n x(t)}{dt^n}$	$(j2\pi f)^n X(f)$
Integral	$\int_{-\infty}^{\infty} x(t) dt$	$\frac{1}{j2\pi f}X(f) + \frac{1}{2}X(0)\delta(f)$
	$\frac{1}{-j2\pi} \frac{d}{dt} X(f)$	$tx(t)$
Convolution	$x_1(t) * x_2(t)$	$X_1(f)X_2(f)$
Conjugation	$x^*(t)$	$X^*(f)$
Multiplication	$x_1(t) * x_2(t)$	$X_1(f) * X_2(f)$

System Analysis with Fourier transform

By means of Fourier transform the differential equations of a LTI system is transformed into algebraic equations, simplifying the analysis. As you know convolution in time domain corresponds to multiplication in frequency domain, i.e.,

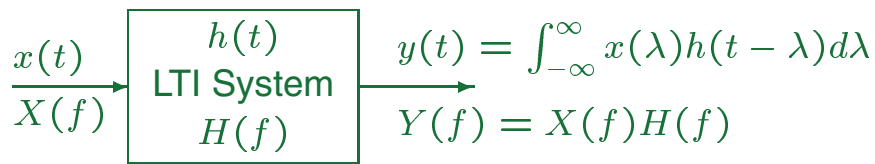
$$x_1(t) * x_2(t) \xleftrightarrow{\mathcal{F}} X_1(f)X_2(f)$$

if $h(t)$ is the impulse response of a system, the response to the input signal $x(t)$ is $y(t) = h(t) * x(t)$, or in frequency domain

$$Y(f) = H(f)X(f)$$

where $H(f) = |H(f)|\angle H(f)$ is referred to the transfer function of the system and is identical to the frequency response transfer function $H(\omega)$, defined earlier. $|H(f)|$ is the amplitude response function, and $\angle H(f)$ is the phase response function. For real time function $|H(f)| = |H(-f)|$, and $\angle H(f) = -\angle H(-f)$

The response of a LTI system is represented as



The time domain response is the inverse Fourier transform of $Y(f)$, i.e.,

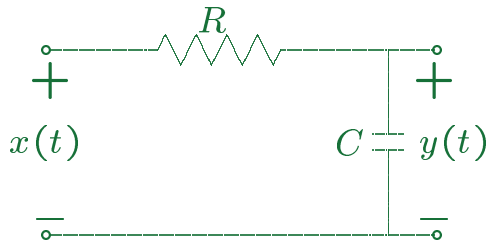
$$y(t) = \int_{-\infty}^{\infty} X(f)H(f)e^{j2\pi ft}df$$

The system transfer function $H(f)$ can be found in following ways:

1. From the impulse response $h(t)$, find $H(f) = \mathcal{F}[h(t)]$.
2. Obtain Fourier transform of the system differential equation and find the ratio $\frac{Y(f)}{X(f)}$ to find $H(f)$.
3. Transform the time domain equations into the phasor domain, expressing circuit elements with impedances and find the transfer function.
4. Find the Laplace transform of the differential equations, obtain the s-domain transfer function and replace s by $j2\pi f$.

Example 4.19 (Example 4-16 Textbook)

Obtain the transfer function of the lowpass RC filter shown



From KVL, we have

$$RC \frac{dy}{dt} + y(t) = x(t)$$

Taking Fourier transform of both side, we have

$$(j2\pi f RC + 1)Y(f) = X(f)$$

$$\begin{aligned} H(f) &= \frac{Y(f)}{X(f)} = \frac{1}{1 + j2\pi f RC} = \frac{1}{1 + j\frac{f}{f_c}} \\ &= \frac{1}{\sqrt{1 + \left(\frac{f}{f_c}\right)^2}} \angle \tan^{-1}\left(\frac{f}{f_c}\right) \end{aligned}$$

where $f_c = \frac{1}{2\pi RC}$ is known as the corner frequency, the half-power frequency, or the 3-dB frequency.

Using the Fourier series method, find the response of the circuit in Example 4.19 when the input is

$$x(t) = Ae^{-\alpha t}u(t)$$

$$X(f) = \mathcal{F}[x(t)] = \frac{A}{\alpha + j2\pi f}$$

The FT of the output is

$$\begin{aligned} Y(f) &= H(f)X(f) = \left(\frac{1}{1 + j2\pi fRC} \right) \left(\frac{A}{\alpha + j2\pi f} \right) \\ &= \frac{A}{RC} \left[\frac{1}{\left(\frac{1}{RC} + j2\pi f\right)(\alpha + j2\pi f)} \right] \\ &= \frac{A}{RC} \left[\frac{k_1}{\frac{1}{RC} + j2\pi f} + \frac{k_2}{\alpha + j2\pi f} \right] \\ &= \frac{A}{RC} \left[\frac{\frac{1}{\alpha - 1/RC}}{\frac{1}{RC} + j2\pi f} + \frac{\frac{1}{1/RC - \alpha}}{\alpha + j2\pi f} \right] \\ &= \frac{A}{\alpha RC - 1} \left[\frac{1}{\frac{1}{RC} + j2\pi f} + \frac{1}{\alpha + j2\pi f} \right] \end{aligned}$$

Finding the inverse Fourier transform, we get

$$y(t) = \frac{A}{\alpha RC - 1} \left[e^{-\frac{t}{RC}} - e^{-\alpha t} \right] u(t)$$

For the case when $\alpha = \frac{1}{RC}$, we may write

$$\begin{aligned} Y(f) &= \left(\frac{1}{1 + j2\pi f RC} \right) \left(\frac{A}{\frac{1}{RC} + j2\pi f} \right) \\ &= \frac{\frac{A}{RC}}{(1 + j2\pi f RC)^2} \end{aligned}$$

Taking the Fourier transform, we get

$$y(t) = \frac{A}{RC} t e^{-\frac{t}{RC}} u(t)$$

Steady-State Response to Harmonic Inputs

The Fourier transform of the output of LTI system is

$$Y(f) = H(f)X(f)$$

If the input $x(t)$ is represented by its Fourier series

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{j2\pi n f_0 t}$$

Using the transform pair $e^{j2\pi n f_0 t} \xleftrightarrow{\mathcal{F}} \delta(f - n f_0)$,

$$X(f) = \sum_{n=-\infty}^{\infty} X_n \delta(f - n f_0)$$

Thus, the Fourier transform of the output is

$$\begin{aligned} Y(f) &= \sum_{n=-\infty}^{\infty} X_n H(n f_0) \delta(f - n f_0) \\ &= \sum_{n=-\infty}^{\infty} |X_n| |H(n f_0)| e^{j(\angle X_n + \angle H(n f_0))} \delta(f - n f_0) \end{aligned}$$

The inverse Fourier transform is

$$y(t) = \sum_{n=-\infty}^{\infty} |X_n| |H(n f_0)| e^{j(2\pi n f_0 t + \angle X_n + \angle H(n f_0))}$$

We conclude that for harmonic inputs at different frequencies, we evaluate $H(\omega)$ and the time domain output at each input frequency and superpose the result.

Example 4.20 (Example 4-18 Textbook)

Consider an ideal lowpass filter with amplitude and phase response function given by

$$|H(f)| = K \Pi \left(\frac{f}{2B} \right) = \begin{cases} K & -B \leq f \leq B \\ 0 & \text{otherwise} \end{cases}$$

and

$$\angle H(f) = -2\pi t_0 f$$

Find the output response to $x(t) = A \cos(2\pi f_0 t + \theta_0)$

$$X_1 = X_{-1}^* = \frac{1}{2} A e^{j\theta_0} \quad \text{with all other } X_n \text{'s} = 0$$

Therefore, the output is

$$y(t) = \begin{cases} 0 & f_0 > B \\ K A \cos[2\pi f_0(t - t_0) + \theta_0] & f_0 \leq B \end{cases}$$

Thus an ideal lowpass filter completely rejects all spectral components with frequencies greater than some cutoff frequency B , and passes all input spectral components below this cutoff frequency. The amplitude is multiplied by a constant K and they are phase shifted by $-2\pi f_0 t_0$ or delayed in time by t_0 .

Bandwidth and Rise Time

A practical measure of bandwidth in frequency domain is the half-power bandwidth, i.e., the range of frequencies over which the magnitude exceeds $\sqrt{1/2}$ times its maximum value. The swiftness of the time-domain response is expressed in terms of the response rise time. We define rise time T_R as the time required for the response to rise from 10 percent to 90 percent of the final value. For a given system, an approximate relation can be found as

$$\omega_B T_R \cong \text{constant}$$

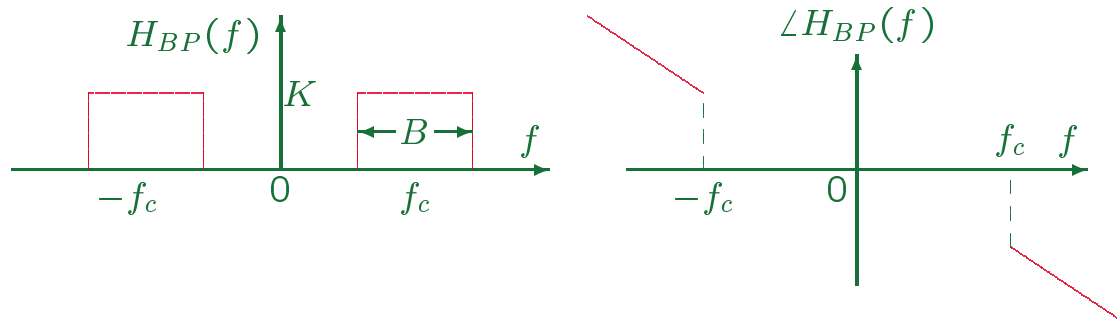
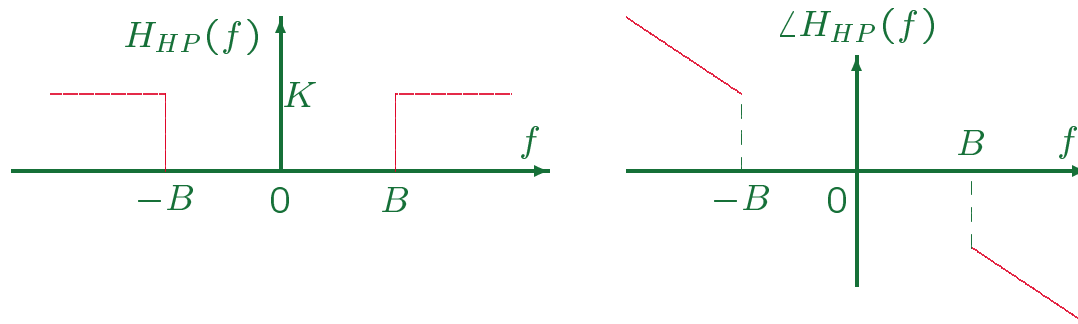
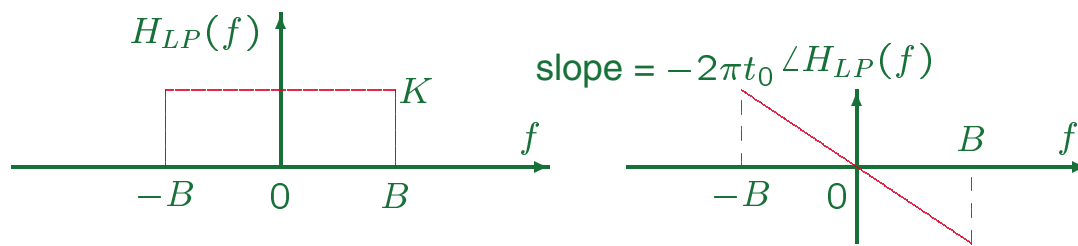
Therefore, for larger the bandwidth the time-domain response rise time is shorter and we have a swifter response.

Ideal Filters

A filter commonly used in electronic signal processing is a frequency-selective device that allows us to shape the magnitude and phase response in a prescribed manner. The transfer function of a filter is written as

$$H(f) = \frac{Y(f)}{X(f)}$$

Where $X(f)$, and $Y(f)$ are the Fourier transform of input and output signals respectively. In this section we consider ideal filters which are not physically realizable, but their concept are useful in the initial stages of system analysis and design. An ideal filter has a transfer function which has a constant amplitude within the pass band and zero elsewhere. The frequency response characteristics of three types of filters known as *ideal lowpass filter*, *ideal high-pass filter*, and *ideal bandpass filter* are shown below.



A distortionless filter permits a linear phase shift and constant gain over the pass band. For example for the ideal low pass filter

$$H_{LP}(f) = K e^{-j2\pi f t_0}$$

where $2\pi f t_0$ is the phase shift. The impulse response of any filter is given by the inverse Fourier transform of its transfer function. The impulse response of the ideal filter is

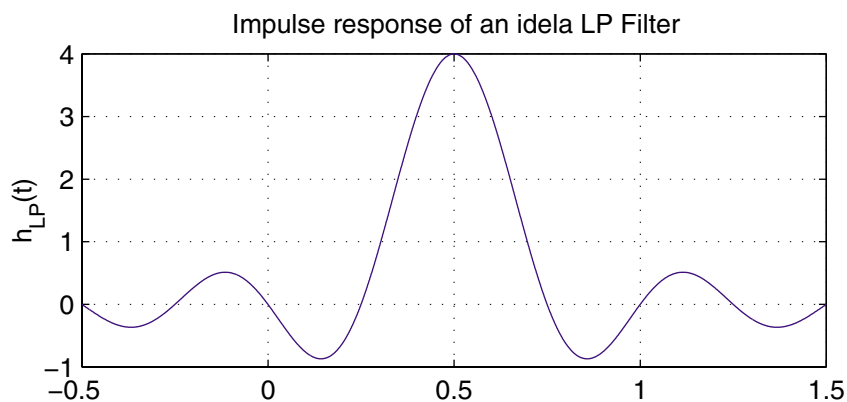
$$\begin{aligned} h_{LP}(t) &= \int_{-\infty}^{\infty} H_{LP}(f) e^{j2\pi f t} df \\ &= \int_{-B}^B K e^{-j2\pi f t_0} e^{j2\pi f t} df \\ &= \int_{-B}^B K e^{j2\pi f (t-t_0)} df \\ &= 2BK \operatorname{sinc}[2B(t-t_0)] \end{aligned}$$

The impulse response $h_{LP}(t)$ for $B = 2$, $t_0 = 0.5$ is obtained as shown.

```

B = 2; K=1; t_0 = .5;
t = -.5:.001:1.5;
hLP = 2*B*K*sinc(2*B*(t - t_0));
plot(t, hLP), grid
xlabel('t'), ylabel('h_{LP}(t)')
title('Impulse response of an idela LP Filter')

```



We see that the impulse response starts long before the impulse occurs at $t = 0$. This suggests that ideal filters are noncausal and cannot be physically realized.

Non-Ideal Filters

The ideal filters are not realizable in practice since they are not causal. In real applications it is necessary to use causal filters. For example the simple RC low-pass filter is a crude approximation of the ideal low-pass filter. In designing analog filters we try to achieve a good enough approximation to the ideal filter such that the frequency response is flat in the pass band and roll-off sharply at the cutoff frequency. One of the filter design commonly used for a low pass filter is the *Butterworth filter*. The amplitude of the FRTF of the RC lowpass filter presented earlier is

$$H(\omega) = \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_c}\right)^2}}$$

We shall now consider the higher order Butterworth filters.

Butterworth Filters

The second-order transfer function in the standard form is

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

For $s = j\omega$, the frequency response transfer function is

$$H(\omega) = \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + j2\zeta\frac{\omega}{\omega_n}}$$

The amplitude is

$$|H(\omega)| = \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\frac{\omega}{\omega_n}\right]^2}}$$

Taking derivative of the $|H(\omega)|$ with respect to ω , we find that the peak of $|H(\omega)|$ occurs at

$$\omega_r = \omega_n \sqrt{1 - 2\zeta^2} \quad \text{for } \zeta < \frac{1}{\sqrt{2}}$$

The system is a lowpass filter when $\zeta \geq \frac{1}{\sqrt{2}}$.

If $\zeta = \frac{1}{\sqrt{2}}$, the resulting lowpass filter is said to be *maximally* flat, since the variation in the magnitude $|H(\omega)|$ is as small as possible across the passband of the filter. Substituting for $\zeta = \frac{1}{\sqrt{2}}$, the cutoff frequency becomes $\omega_c = \omega_n$, and the amplitude becomes

$$|H(\omega)| = \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_c}\right)^4}}$$

This filter is called the second-order Butterworth filter.

Example 4.21

Find the amplitude response of the second order Butterworth filter given by

$$H(s) = \frac{\omega_c^2}{s^2 + \sqrt{2}\omega_c s + \omega_c^2}$$

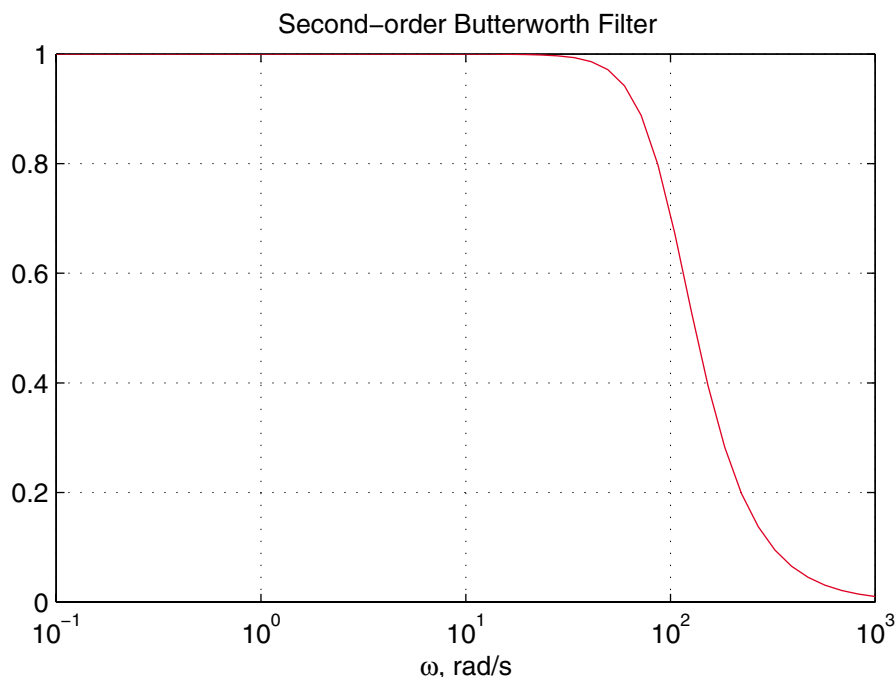
for $\omega_c = 100$ radian/s.

We use the following commands:

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```
wc =100;  
num=wc^2;  
den  =[1 sqrt(2)*wc  wc^2];  
w= logspace(-1, 3);  
[mag, ph] = bode(num, den , w);  
semilogx(w, abs(mag), 'r'), grid  
title('Second-order Butterworth Filter')  
xlabel('\omega, rad/s')
```

The result is



N-pole Butterworth Filter

The Butterworth filter has a maximally flat frequency response across the pass band. The filter transfer function has poles that are equally spaced on a circle of radius equal to the cutoff frequency ω_c in the left half s-plane. For example, the transfer function of the third-order Butterworth filter is

$$H(s) = \frac{\omega_c^3}{(s + \omega_c)(s^2 + \omega_c s + \omega_c^2)} = \frac{\omega_c^3}{s^3 + 2\omega_c s^2 + 2\omega_c^2 s + \omega_c^3}$$

For $s = j\omega$, the amplitude of the third-order filter is

$$|H(\omega)| = \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_c}\right)^6}}$$

In general, the magnitude of the frequency response function for the Butterworth filter is

$$|H(\omega)| = \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_c}\right)^{2n}}}$$

where n is the order of the filter, and for the passive filters n also indicates the number of reactive elements. For any value of n , the cutoff frequency $\omega_c = 1$ rad/sec. For large ω the amplitude response approximate to $|H(\omega)| = 1/\omega^n$. That is, the high frequency asymptote roll-off at a slope of $-20n$ dB/decade.

The transfer function can be determined from a table of the Butterworth polynomials. The Signal Processing Toolbox function `[z, p, k] = buttap(n)` returns the zero, poles and the gain of a normalized n order low-pass Butterworth filter.

Example 4.22

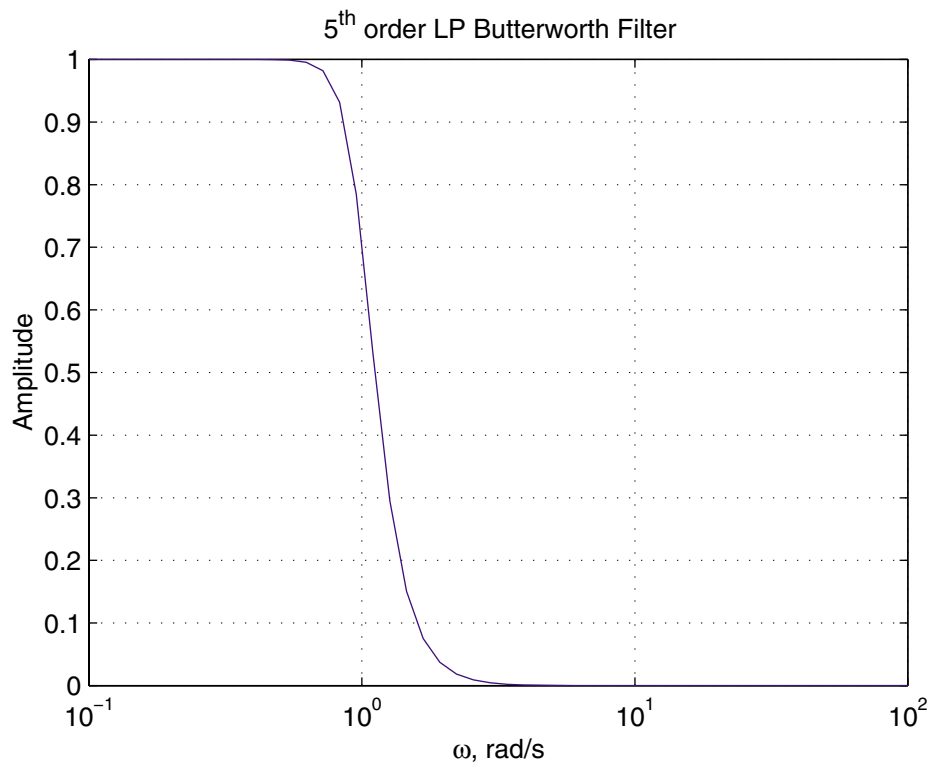
Design a normalized 5th order lowpass Butterworth filter and plot the frequency response amplitude.

We use the following commands

```
[z, p, k] = buttap(5) % Returns zeros, poles, and gain.
[num,den]=zp2tf(z,p,k)% transforms zero and poles to
                    % polynomial Transfer function
w= logspace(-1, 2);
[mag, phase] = bode(num, den, w);
semilogx(w, mag), grid
xlabel('\omega, rad/s'), ylabel('Amplitude')
title('5th order LP Butterworth Filter')
```

The result is

```
z = []
p = -0.3090 + 0.9511i
    -0.3090 - 0.9511i
    -0.8090 + 0.5878i
    -0.8090 - 0.5878i
    -1.0000
k = 1
num= 0 0 0 0 0 1  den= 1 3.2361 5.2361 5.2361 3.2361 1
```



Thus, the fifth order lowpass Butterworth filter transfer function is

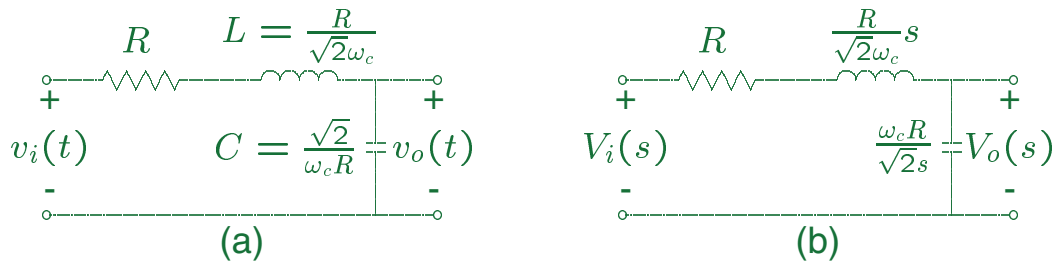
$$H(s) = \frac{1}{s^5 + 3.2361s^4 + 5.2361s^3 + 5.2361s^2 + 3.2361s + 1}$$

Filter Realization

For a given type of filter first the transfer function is obtained based on the specification on the desired bandwidth and the stopband attenuation. Then using network synthesis the filter is designed with passive elements and op amps. You learn how to realize various types of filters in EE-404 (Active Filters) and EE-422 (Digital signal Processing). The following circuits shows RLC realization of second- and third-order Butterworth filters.

RLC Realization of a Second-order Butterworth Filter

Consider the RLC circuit shown in (a) and its s-domain equivalent circuit in (b).



Applying the voltage divider concept to the s-domain circuit in (b), we get

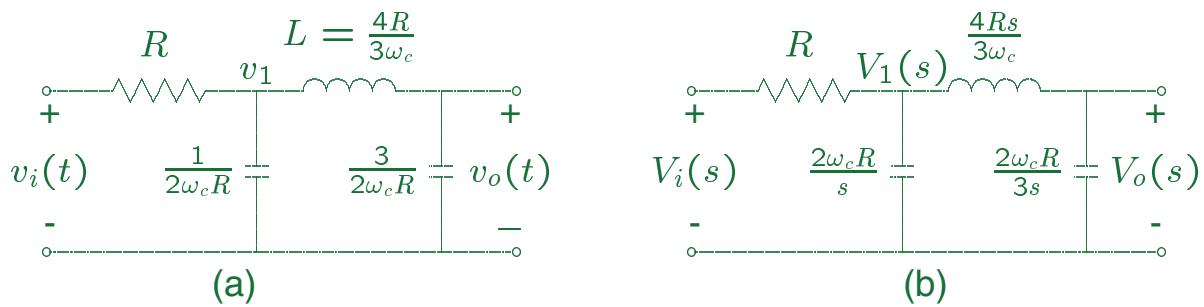
$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{\frac{\omega_c R}{\sqrt{2} s}}{\frac{R}{\sqrt{2} \omega_c} s + R + \frac{\omega_c R}{\sqrt{2} s}}$$

Multiplying numerator and denominator by $\frac{\sqrt{2} \omega_c s}{R}$ will result in the second-order Butterworth filter

$$H(s) = \frac{\omega_c^2}{s^2 + \sqrt{2} \omega_c s + \omega_c^2}$$

RLC Realization of a third-order Butterworth Filter

Consider the RLC circuit shown in (a) and its s-domain equivalent circuit in (b).



Writing the s-domain node voltage equations for circuit (b), we obtain

$$\begin{bmatrix} \frac{s}{2\omega_c R} + \frac{1}{R} + \frac{3\omega_c}{4Rs} & -\frac{3\omega_c}{4Rs} \\ -\frac{3\omega_c}{4Rs} & \frac{3s}{2\omega_c R} + \frac{3\omega_c}{4Rs} \end{bmatrix} \begin{bmatrix} V_1(s) \\ V_o(s) \end{bmatrix} = \begin{bmatrix} \frac{1}{R} \\ 0 \end{bmatrix} V_i$$

Using Cramer's rule to solve for $V_2(s)$, the determinants are

$$\Delta = \frac{3}{4\omega_c^2 R^2 s} (s^3 + 2\omega_c s^2 + 2\omega_c^2 s + \omega_c^3)$$

$$\Delta_2 = \frac{3\omega_c}{4R^2 s} V_i(s)$$

The output voltage is given by

$$V_o(s) = \frac{\Delta_2}{\Delta}$$

Substituting for Δ_2 and Δ , we obtain the transfer function for the third-order Butterworth filter

$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{\omega_c^3}{s^3 + 2\omega_c s^2 + 2\omega_c^2 s + \omega_c^3}$$

Many standard Tables are available with a list of components and circuit configuration for each type of filter, thus it is not necessary to synthesize the filter using circuit equations.

The MATLAB Signal Processing Toolbox contains numerous functions for designing lowpass, highpass and bandpass filters for both analog and digital filters. These includes the classical IIR (infinite-impulse-response), Butterworth, Chebyshev type I and II, Elliptic, and Bessel.

Following are some of the functions from the Signal Processing Toolbox for filter design

```
[num, den] = butter(n, wn, 'ftype', 's')  
[num, den] = cheby1(n, Rp, wn 'ftype', 's')  
[num, den ]= cheby2(n, Rp, wn 'ftype', 's')  
[num, den] = besself(n, wn, 'ftype')  
[num, den] = ellip(n, Rp, , Rs, wn 'ftype', 's')
```

For their use and the description of function arguments type **help** followed by the function name at the MATLAB prompt. The Toolbox includes many other functions for FIR (finite impulse response) filters design. The filter design and synthesis is beyond the scope of this course, Use of Butterworth function for few filter design is described below:

The command `[num, den] = butter(n, wn, 's')` designs an order **n** lowpass analog Butterworth filter with cutoff frequency **wn**. **s** specifies that we have an analog filter and require its s-domain function. If **wn** is a two element vector `[w1 w2]` with **w1** < **w2**, the function returns an order **2n** bandpass analog filter with bandpass **w1** < **w** < **w2**.

The command `[num, den] = butter(n, wn, 'ftype','s')` designs a highpass or bandstop filter, where type is **high** for highpass analog filter with cutoff frequency **wn**.

stop for an order **2n** bandstop analog filter if **wn** is a two-element vector `[w1, w2]`.

All functions return the numerator and denominator coefficients of the transfer function when the output is specified with two arguments `[num, den]`. If three arguments `[z, p, k]` are specified, the functions return zeros, poles and gain of the transfer function. Finally for four arguments `[A, B, C, D]`, the functions return the state-space models.

Example 4.23

Consider the design of a Butterworth lowpass filter to meet the following specifications:

Bandwidth $\omega_c = 100$ rad/s, with the frequency response function amplitude of -60dB at $\omega = 1000$ rad/s.

$$20\log |H(\omega)| = -60 \quad \text{or} \quad |H(\omega)| = 10^{-3}$$

The magnitude of the frequency response function for the Butterworth filter is

$$|H(\omega)| = \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_c}\right)^{2n}}}$$

Therefore, we have the magnitude of the frequency response function for the Butterworth filter is

$$|H(\omega)| = 10^{-3} = \frac{1}{\sqrt{1 + \left(\frac{1000}{100}\right)^{2n}}}$$

or

$$10^6 = 1 + (10)^{2n}$$

Neglecting 1, we get $2n = 6$, or $n = 3$.

We use the following commands to obtain the transfer function of the third-order Butterworth filter and to obtain its frequency response.

```
[num, den]=butter(3, 100, 's') % Returns num, and den  
                                % of the 3rd order Butterworth lowpass filter  
bode(num, den)
```

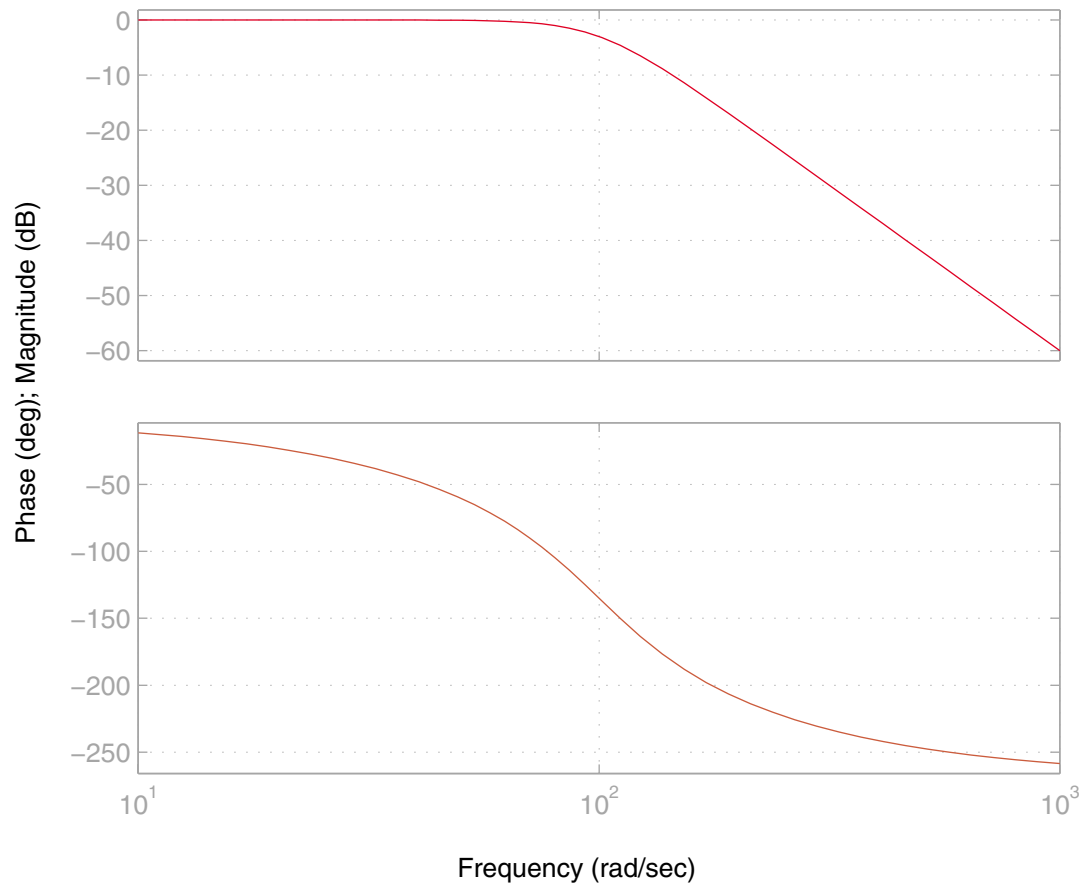
The result is

```
num =  
      0          0          0          1.0000e+006  
den =  
      1.0000e+000  2.0000e+002  2.0000e+004  1.0000e+006
```

The transfer function is

$$H(s) = \frac{(100)^3}{s^3 + 2(100)s^2 + 2(100)^2s + (100)^3}$$

Bode Diagrams



Example 4.24

Design a third-order highpass Butterworth filter with cutoff frequency of 1000 rad/sec. and obtain the frequency response.

The following commands:

```
[num, den] = butter(3, 1000, 'high', 's')  
bode(num, den)
```

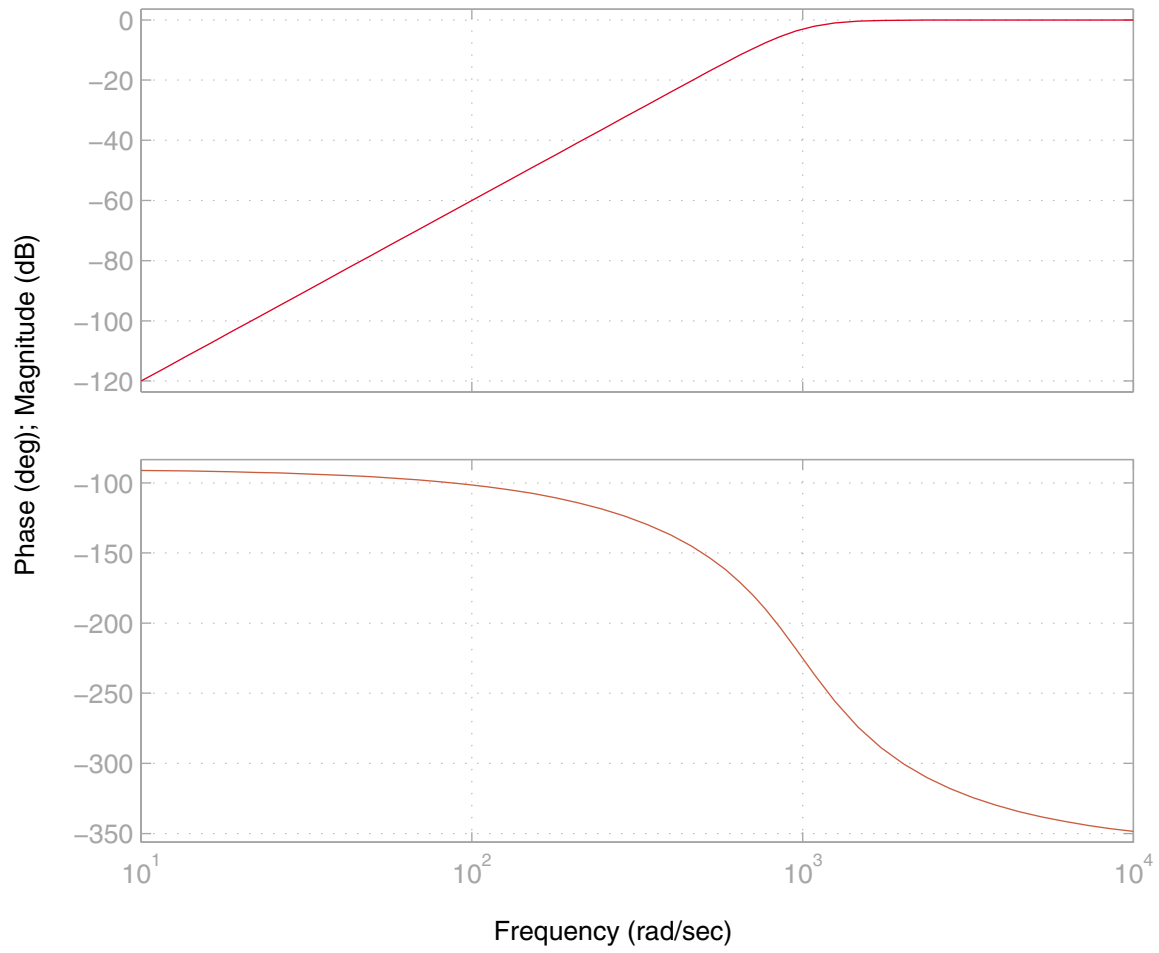
result in

```
num =  
      1          0          0          0  
den =  
      1.00e+000  2.00e+003  2.00e+006  1.00e+009
```

From the above result the filter transfer function is

$$H(s) = \frac{s^3}{s^3 + 2(1000)s^2 + 2(1000)^2s + (1000)^3}$$

Bode Diagrams



Example 4.25

(a) Design a fourth-order bandpass Butterworth filter with a passband from 300 to 500 rad/sec, and obtain the frequency response.

(b) The following signal is applied to the filter

$$x(t) = \sin(50t) + \sin(400t) + \sin(1200t)$$

Find the attenuation and phase shift introduced by the filter for each component of the input signal. Determine the expression for the output signal and obtain a plot of input and output signals.

The following commands:

```
[num, den] = butter(2, [300 500], 's')
bode(num, den)
w=[50 400 1200];
[H, Adeg]= bode(num, den, w); % |H(w)| & angle
A = Adeg*pi/180; % Phase angle in radian
disp([' |H| Angle, rad'])
disp([H, A]), t=0:.03/500:.03;
x = sin(50*t)+sin(400*t)+sin(1200*t));
y = H(1)*sin(50*t+ A(1)) + H(2)*sin(400*t+A(2))...
+ H(3)*sin(1200*t+ A(3));
figure(2), plot(t,x, t, y), xlabel('t, sec')
text(.001,1.75,'x(t)'), text(.0036, 1.15,'y(t)')
```


result in

```
num =  
      0          0          4.0000e+004  0          0  
den =  
      1.0000e+000  2.8284e+002  3.4000e+005  4.2426e+007  
      2.2500e+010
```

H	Angle, rad
0.0046	3.0456
0.9999	6.1055
0.0346	3.4077

From the above result, the filter transfer function is

$$H(s) = \frac{40000s^2}{s^4 + 282.84s^3 + 340000s^2 + 42.426 \times 10^6 + 2.25 \times 10^{10}}$$

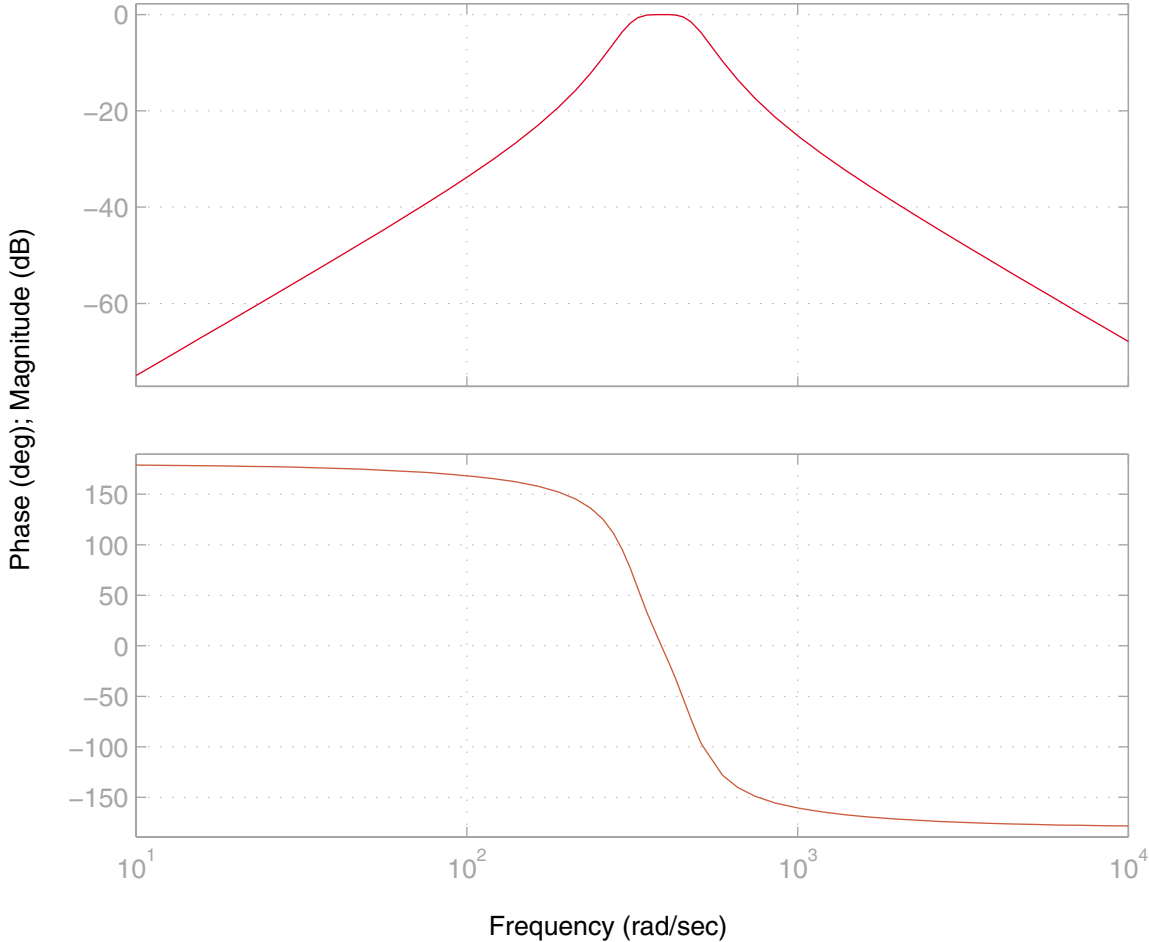
As we can see from the result for $H(\omega)$, the 50 and 1200 rad/sec components are attenuated greatly, and the output signal is approximately

$$y(t) \simeq \sin 400t$$

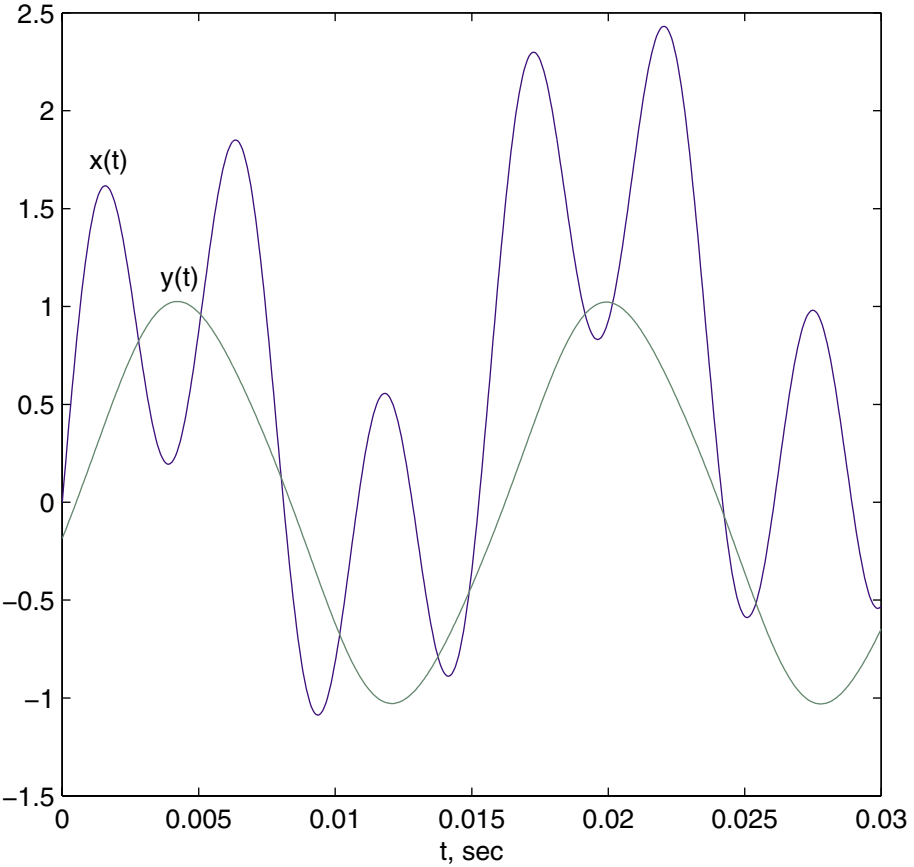
The frequency response, and the input and output signals are as shown.

Frequency response for the filter in Example 4.25.

Bode Diagrams



Input and output signals of Example 4.25.



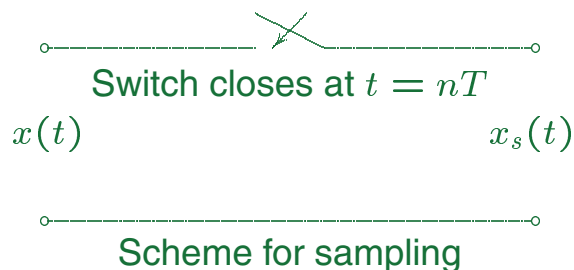
Interactive Signal Processing Environment

The signal processing Toolbox includes an interactive graphical user interface (GUI), called **sptool**, for analyzing and manipulating digital signals, filters, and spectra. To open this interactive GUI environment type **sptool** at the MATLAB prompt. Digital signal processing is covered in EE-422. To learn how to use this tool refer to the Signal Processing Toolbox, or you can access the online documentation by selecting **helpdesk** from **help** menu, or typing helpdesk.

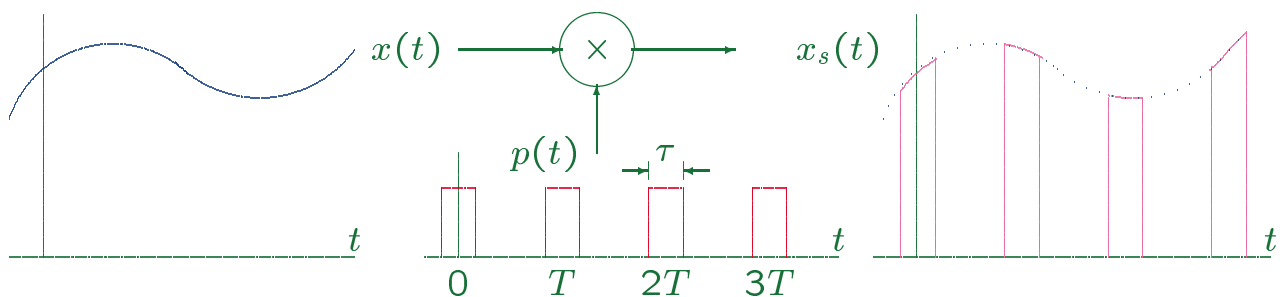
Sampling

Sampling provides the mechanism for converting analog signals to digital signals. The digitized signal can be processed by a digital computer or can be transmitted over a digital communication system.

Sampling of a continuous-time signal $x(t)$ can be illustrated by a rotating switch that is closed for an instant every T seconds as shown.



The action of the switch can be modeled as a periodic pulse train $p(t)$ of unit height, with period T , and pulse duration τ . The sampled signal $x_s(t)$ equals $x(t)$ for the τ seconds that the switch remains closed and is zero when the switch is open.



The sampled signal is given by

$$x_s(t) = p(t)x(t)$$

When designing an electronic system it is very important to know the frequency spectrum of the signal, thus we need to obtain the Fourier transform of the sampled signal. The Fourier transform $P(f)$ is a train of impulses whose strength C_n equal the Fourier series coefficients of $p(t)$, i.e.,

$$P(f) = \sum_{-\infty}^{\infty} C_n \delta(f - nf_s)$$

where

$$C_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} p(t) e^{-jn2\pi f_s t} dt$$

f_s is the fundamental frequency of $p(t)$, which is also the sampling frequency

$$f_s = \frac{1}{T}$$

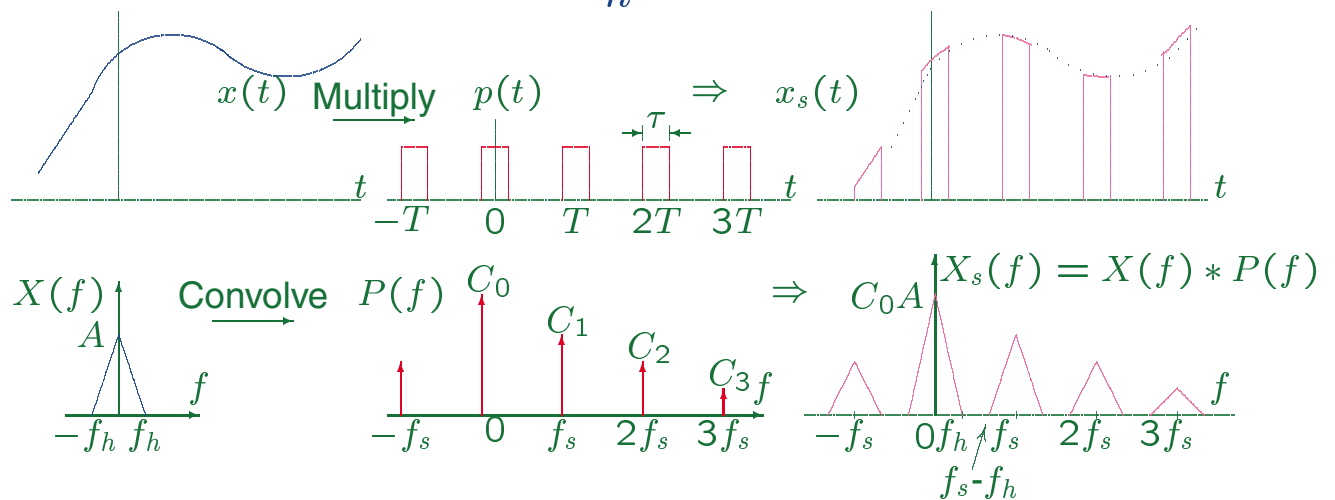
The spectrum $X_s(f)$ of the sampled signal $x_s(t) = x(t)p(t)$ is described by the convolution

$$X_s(f) = X(f) * P(f) = X(f) * \sum_{-\infty}^{\infty} C_n \delta(f - nf_s)$$

or

$$X_s(f) = \sum_{-\infty}^{\infty} C_n X(f - nf_s)$$

Assuming the signal $x(t)$ to be bandlimited, its Fourier transform whose spectrum $X(f)$ is assumed as shown lies in the bandwidth $-f_h < f < f_h$. That is, X_f is assumed zero for $|f| \geq f_h$. The spectrum of the sampled signal $X_s(f)$ is the superposition of the scaled shifted replica of $X(f)$. That is each of the spectra is multiplied by the corresponding Fourier series constant C_n which decay as $\frac{1}{n}$.



If $X(f)$ is to be recoverable from $X_s(f)$, and consequently $x(t)$ from $x_s(t)$, then

$$f_s - f_h \geq f_h \quad \text{or} \quad f_s \geq 2f_h$$

Thus, if there is no spectral overlap, the image about the origin equals $C_0X(f)$, and X_f can be recovered by passing the sampled signal through an ideal low pass filter with a cutoff frequency of $f_h = 0.5f_s$ and a gain of $\frac{1}{C_0}$.

Shannon Sampling Theorem

A bandlimited signal, $x(t)$, having no frequency component above f_h can be reconstructed perfectly from their samples provided the sampling rate $f_s \geq 2f_h$. That is, the time between the samples must be no greater than $\frac{1}{2f_h}$. The critical sampling rate $f_s = 2f_h$ is called the Nyquist rate, and the critical sampling interval $\frac{1}{2f_h}$ is called the Nyquist interval. The Nyquist rate is determined by the bandwidth of the given (bandlimited signal) $x(t)$.

Claude Shannon and Harry Nyquist were researchers at Bell Lab. who made significant contribution to the theory of sampling and digital communication.

Ideal Sampling (Impulse-Train Sampling)

In digital systems, the sampling signal is an extremely narrow pulse having infinitely small width τ . Thus, to simplify the computation of the transform $x(t)p(t)$, the pulse train $p(t)$ is replaced by a train of impulses given by

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$P(f) = \sum_{-\infty}^{\infty} C_n \delta(f - nf_s)$$

The Fourier series coefficients are

$$C_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t) e^{-jn2\pi f_s t} dt = \frac{1}{T} = f_s$$

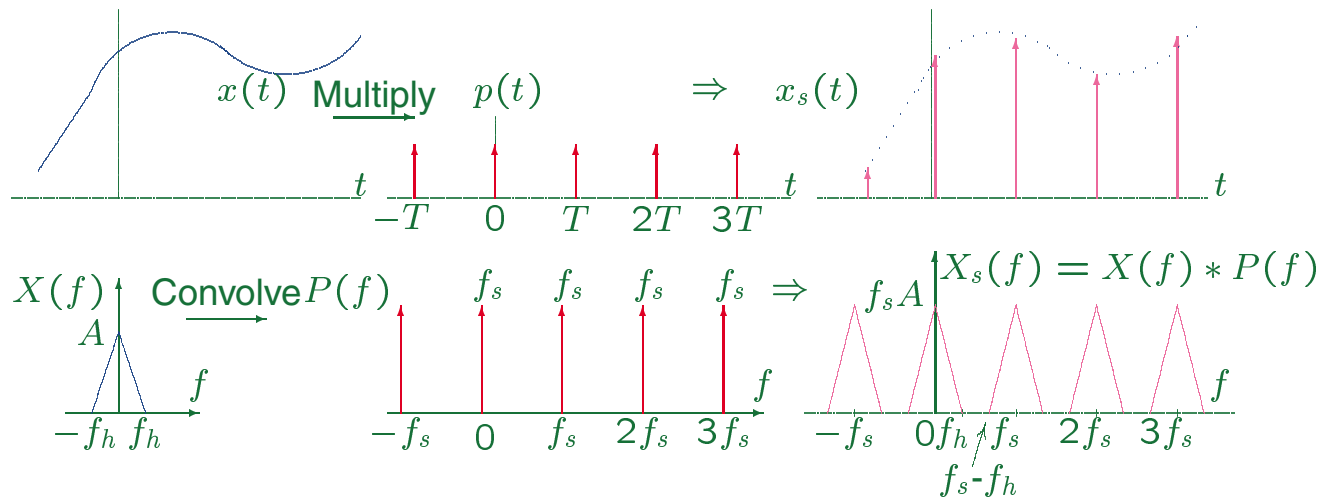
or

$$P(f) = f_s \sum_{-\infty}^{\infty} \delta(f - nf_s)$$

Thus, the spectrum of the sampled signal becomes

$$X_s(f) = f_s \sum_{-\infty}^{\infty} X(f - nf_s)$$

The spectra of the signals for ideal sampling is as shown.



Note that all translated spectra have the same amplitude. Once again X_f and consequently $x(t)$ can be recovered by passing the sampled signal through an ideal low pass filter.

Sampling at a rate slower than the Nyquist rate cause the spectra images to overlap and we cannot recover $x(t)$. The occurrence of spectra overlap is known as *aliasing*.

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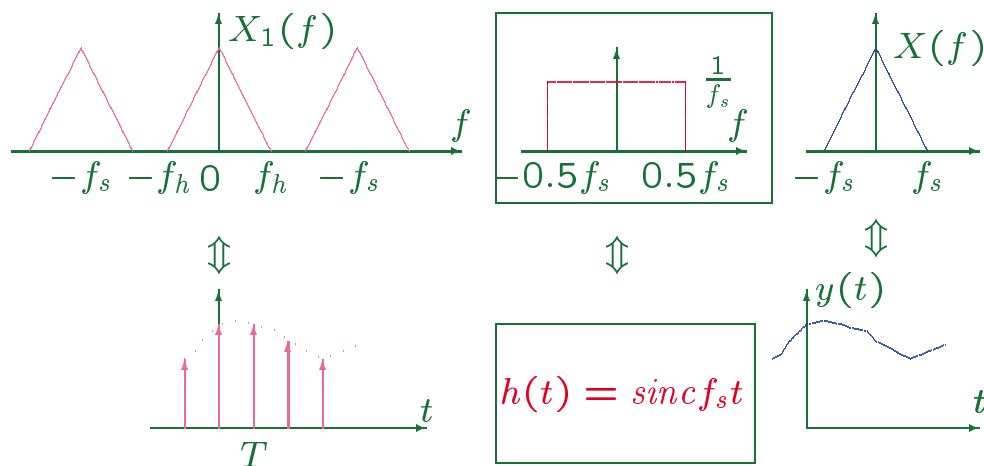
The practical signals are not absolutely bandlimited, and thus reconstruction of the original signal without any error is not possible. However, in all practical signals there is some frequency beyond which the energy is negligible. This frequency is usually taken as the bandwidth. Also, in practice we have to use non-ideal causal lowpass filters which results in some error in the recovered signal. There is a tradeoff between the complexity of the filter design and the accuracy of the reconstructed signal. In order to improve the accuracy sampling is done at a much higher frequency than the Nyquist rate. Oversampling causes the spectra images to be sufficiently far apart so that a more realistic filter can be used. Nonbandlimited signals are usually sampled at approximately 10 times the frequency at which the amplitude spectrum is 3 dB down from its maximum value.

Ideal Signal Recovery and *sinc* Interpolation

For ideal sampling, i.e., when the sampling signal is a train of impulses, the sampled signal is

$$x_s(t) = x(t) \sum_{k=-\infty}^{\infty} \delta(t-kT) = \sum_{k=-\infty}^{\infty} x(kT)\delta(t-kT)$$

We can recover the signal by passing $x_s(t)$ through an ideal lowpass filter with a gain of $\frac{1}{f_s}$ and a cutoff frequency of $0.5f_s$.



The recovered signal $y(t)$ is therefore given by the convolution

$$\begin{aligned} y(t) &= x_s(t) * h(t) = \left[\sum_{k=-\infty}^{\infty} x(kT)\delta(t-kT) \right] * h(t) \\ &= \sum_{k=-\infty}^{\infty} x(kT)h(t-kT) \end{aligned}$$

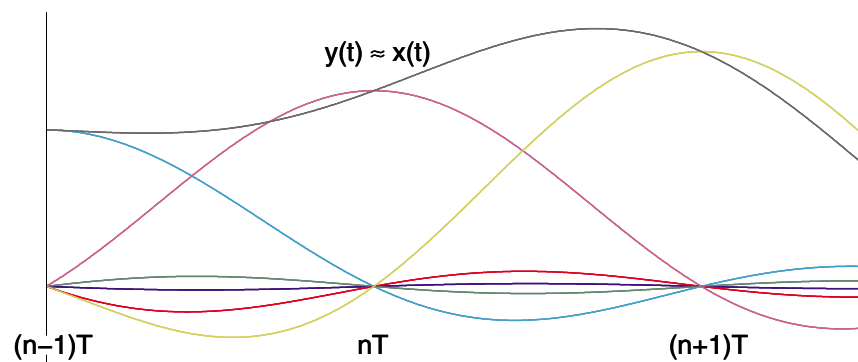
The impulse response of a lowpass filter is a *sinc* function given by

$$h(t) = \text{sinc } f_s t \quad \text{or}$$
$$h(t - kT) = \text{sinc } f_s (t - kT) = \text{sinc} \left(\frac{t}{T} - k \right)$$

Substituting for $h(t)$ we find the following relation known as *interpolation formula*.

$$x(t) = \sum_{k=-\infty}^{\infty} x(kT) \text{sinc} \left(\frac{t}{T} - k \right)$$

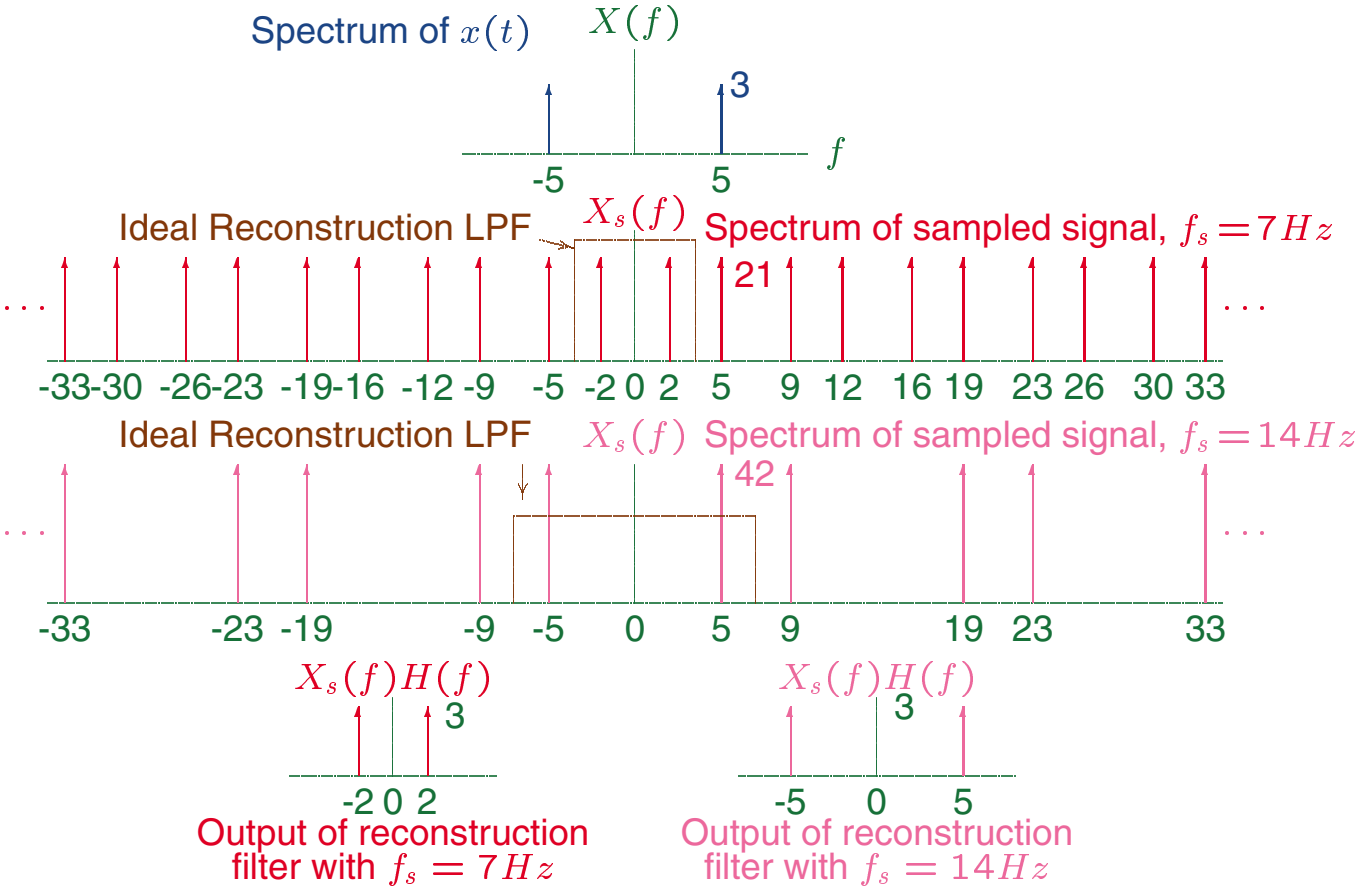
This equation shows that the original signal $x(t)$ can be reconstructed by weighting each sample by a sinc function centered at the sample time and summing as illustrated below.



Example 8.1 (Textbook Example 8-1)

Consider the signal $x(t) = 6 \cos 2\pi(5)t$

Sampled at 7 Hz and 14 Hz. Since the highest frequency (the only frequency in this case) is 5 Hz, we will see the effect of sampling a signal at both a frequency less than and greater than twice the highest frequency in $x(t)$.



The spectrum of the signal is

$$X(f) = 3\delta(f - 5) + 3\delta(f + 5)$$

Thus the spectrum of the sampled signal is

$$X_s(f) = 3f_s \sum_{n=-\infty}^{\infty} \delta(f - 5 - nf_s) + \delta(f + 5 - nf_s)$$

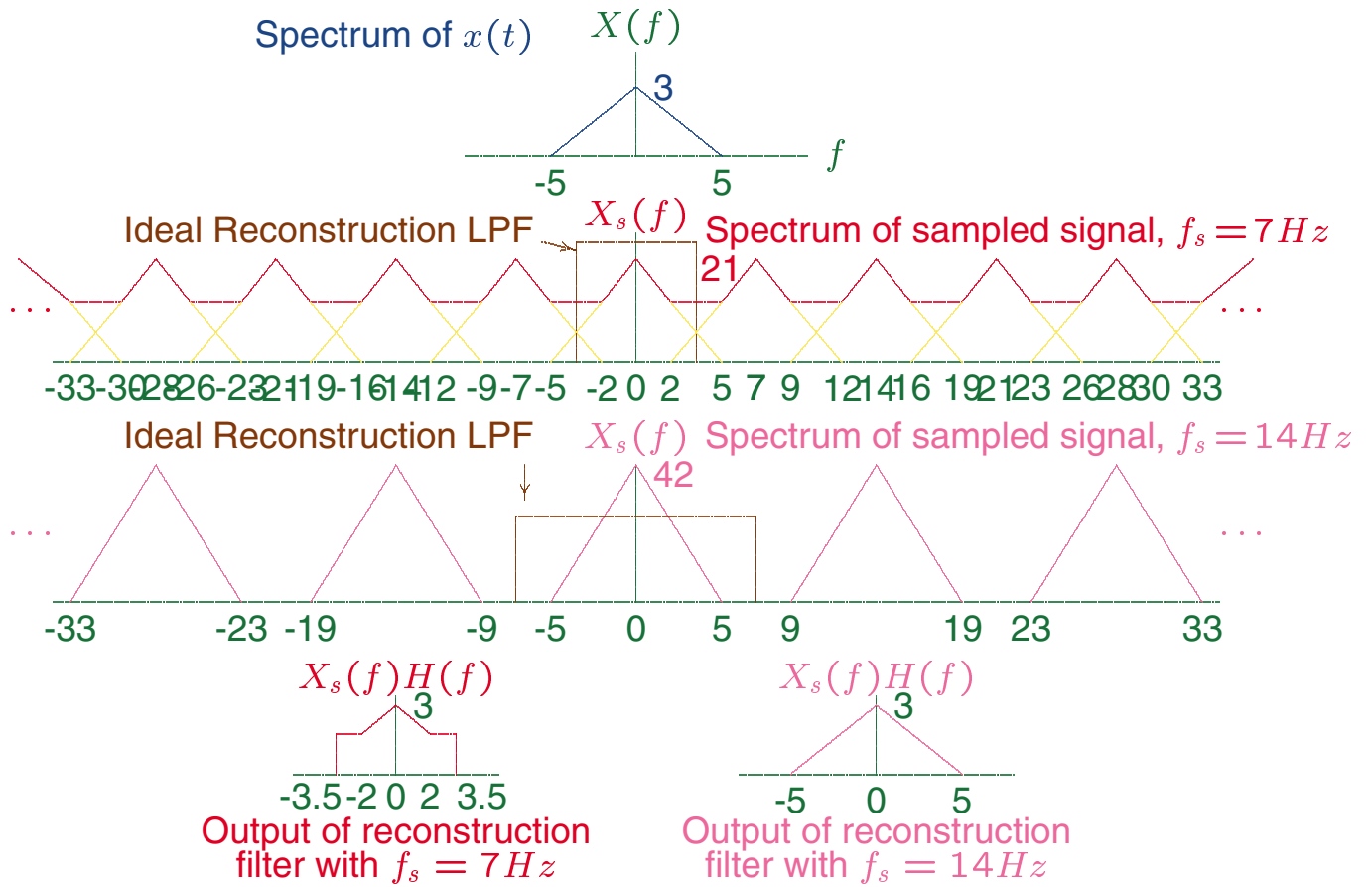
The assumed reconstruction filter, $H(f)$, is an ideal LPF with a bandwidth of $0.5f_s$, and amplitude of $T = \frac{1}{f_s}$. When undersampling (7 Hz) at a frequency less than twice the highest frequency in $x(t)$, which is less than 2×5 Hz, aliasing will occur. That is the reconstructed signal has an incorrect frequency. On the other hand sampling at a frequency greater than twice the highest frequency in $x(t)$ ($14 \text{ Hz} > 2 \times 5 \text{ Hz}$), the reconstructed signal is identical to the original signal.

Example 8.2 (Textbook Example 8-2)

Consider a nonperiodic signal $x(t)$ that has a continuous spectrum $X(f)$ with highest frequency 5 Hz as shown. The signal is sampled at 7 Hz and 14 Hz. Plot the spectrum of the sampled signal for each sampling frequency.

$$X_s(f) = f_s \sum_{-\infty}^{\infty} X(f - nf_s)$$

The sampled spectrum for each sampling frequency is as shown. For sampling at 7 Hz, overlap of the translated spectra (aliasing) occurs and $X_s(f)$ is found by summing spectra in the overlapped regions. For sampling at 14 Hz which exceeds the Nyquist rate (10 Hz), there is no spectral overlap. We have assumed an ideal reconstruction filter with amplitude response $T = \frac{1}{f_s}$, and a bandwidth of $0.5f_s$. The output spectrum for each case is also shown. It is clear that the output of reconstruction filter is identical to the original signal for sampling at 14 Hz.



Interpolation

Interpolation is the process of assessing values between the sampled intervals, and is useful when the sampling rate is increased. This may be in the form of step interpolation (Zero-Order-Hold Sampling), Linear Interpolation, or Sinc Interpolation

Zero-Order-Hold Sampling

In practice, analog signals are sampled using Zero-order-hold devices that acquire a sample value and hold it constant until the next sample is acquired. The sampled signal resemble a staircase that remains flat between the sample points.

Linear Interpolation

Linear Interpolation uses a triangular interpolating function which produces a linear approximation between the sampled values.

Example 8.3

A sampled signal is expressed by $x[n] = [-1, 2, 3, 2]$ for $n = 0, 1, 2, 3$ seconds, The sampling period $T = 1$. Find the value of the reconstructed signal $x(t)$ at 2.5 seconds.

(a) For step interpolation, the signal value at $t = 2.5$ is the value at $t = 2$, i.e., $x(2.5) = 3$ s

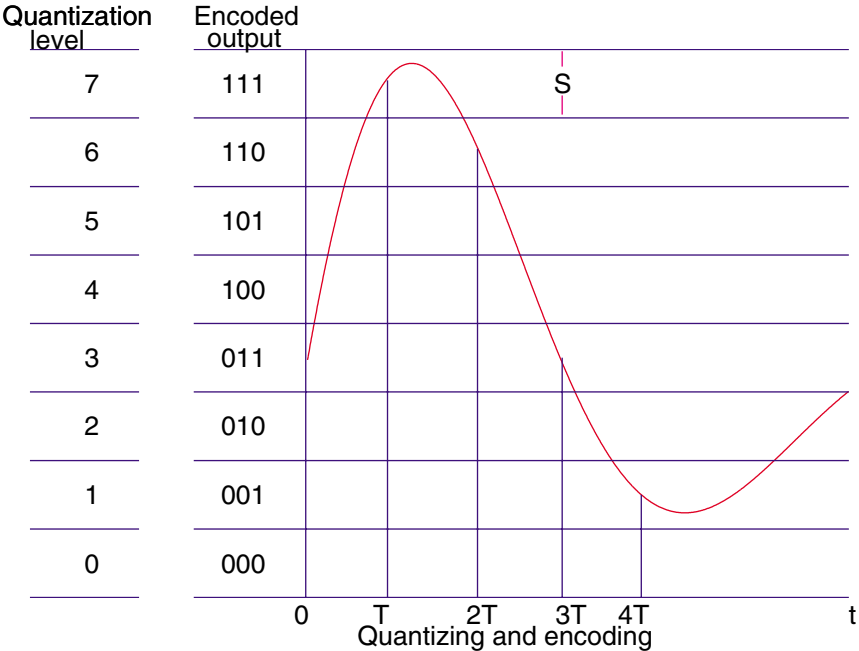
(b) For linear interpolation, the signal value at $t = 2.5$ is the average of the values at $t = 2$ and $t = 3$, i.e., $x(2.5) = 0.5(3 + 2) = 2.5$ s.

Quantization and Encoding

Digital signals are discrete-time signals which are quantized in amplitude by rounding to the nearest level. Encoding is done by representing each of the quantized value by an indexed sequence of word of zero and one bits that can be processed digitally. For a binary representation, the number of quantizing levels q and the digital word length n are related by

$$q = 2^n$$

The quantizing and encoding is illustrated below. where S represents the width of the quantization interval.



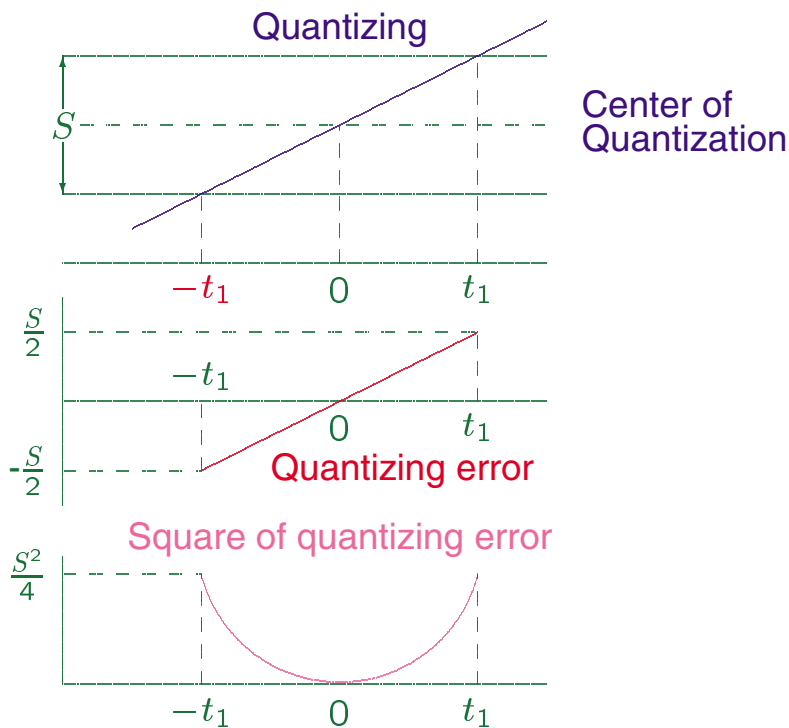
Quantization Error

When the sampled values are quantized errors are introduced in the process. No matter how fine the quantization, its effects are irreversible, since the world length must necessarily be finite. Thus, analyzing the errors introduced by quantization is important to the study of practical DSP systems.

If the quantizing level is at the center of the interval, the maximum value of the quantization error lies between $\pm \frac{S}{2}$, where S is the width of the quantizing interval. For a very large number of quantizing intervals S will be sufficiently small and the signal $x(t)$ will be nearly linear within the interval as shown. The error $\epsilon(t)$ introduced by assuming $x(t)$ to remain in the quantizing interval $2t_1$ is

$$\epsilon(t) = \frac{S}{2t_1}t$$

as shown.



The mean square error E is given by

$$E = \frac{1}{2t_1} \int_{-t_1}^{t_1} \epsilon^2(\alpha) d\alpha = \frac{1}{t_1} \int_0^{t_1} \epsilon^2(\alpha) d\alpha$$

or

$$E = \frac{1}{t_1} \int_0^{t_1} \left(\frac{S}{2t_1} \right)^2 \alpha^2 d\alpha = \frac{S^2}{12}$$

E has the dimension of Watts and is regarded as noise power. Assuming D is the maximum variation of $x(t)$, we have

$$D = \max[x(t)] - \min[x(t)]$$

Since there are $q = 2^n$ quantizing level, the quantizing step size is

$$S = \frac{D}{2^n} = D2^{-2n}$$

Substituting in E , we get

$$E = \frac{D^2}{12} 2^{-2n}$$

It is clear that the mean-square error decreases exponentially with the A/D converter wordlength, n . It is customary to define the signal-to-noise ratio SNR as

$$\text{SNR} = \frac{P_s}{E}$$

where P_s is the signal power.

Substituting for E , we have

$$\text{SNR} = \frac{P_s}{(D^2/12)2^{-2n}} = 12P_s D^{-2} 2^{2n}$$

The SNR is usually expressed in decibels, i.e.,

$$\text{SNR}_{dB} = 10 \log_{10}(12) + 10 \log_{10} P_s - 20 \log_{10} D + 20n \log_{10}(2)$$

or

$$\text{SNR}_{dB} = 10.79 + 6.02n + 10 \log_{10} P_s - 20 \log_{10} D$$

This result suggests a 6-dB improvement in SNR for each additional bit.

Example 8.4

A sampled signal that varies between $-2V$ and $2V$ is quantized using n bits. What value of n will ensure an rms quantization error of less than 5 mV .

The rms quantization error is $\sqrt{E} = 5 \times 10^{-3}$ or $E = 25 \times 10^{-6}$.

The range $D = 4V$. The mean-square error is given by

$$E = \frac{D^2}{12} 2^{-2n}$$

or

$$2^{2n} = \frac{D^2}{12E} = \frac{16}{(12)(25 \times 10^{-6})} = 53333.33$$

$$\text{or } 2^n = 230.94$$

$$n = \log_2(230.94) = \frac{\ln(230.94)}{\ln(2)} = 7.85$$

Rounding up, we get $n = 8$ bit

Example 8.5 (Textbook Example 8-3)

A sinusoidal signal $x(t) = A \cos \omega t$ is sampled and quantized. Find the SNR in decibels

The signal power of a sinusoid is

$$P_s = \frac{1}{2}A^2$$

$$10 \log_{10} P_s = 10 \log_{10} \frac{A^2}{2} = 20 \log_{10} A - 3.01$$

The dynamic range D is the peak-to-peak value of the signal, i.e.,

$$D = 2A \quad \text{and}$$

$$20 \log_{10} D = 20 \log_{10} 2A = 6.02 + 20 \log_{10} A$$

The SNR in decibels is given by

$$\begin{aligned} \text{SNR}_{dB} &= 10.79 + 6.02n + 10 \log_{10} P_s - 20 \log_{10} D \\ &= 10.79 + 6.02n + 20 \log_{10} A - 3.01 - 6.02 - 20 \log_{10} A \end{aligned}$$

or

$$\text{SNR}_{dB} = 1.76 + 6.02n$$

Discrete Fourier Transform

Discrete Fourier transform (DFT) is the mapping of a set of N discrete-time samples into the frequency domain having N harmonics. This results in a finite set of discrete-frequency spectrum.

We found the Fourier transform of a continuous-time nonperiodic signal as

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft} dt$$

and its inverse Fourier transform as

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft} df$$

We now consider a time-limited signal with a finite time interval $[0, T]$. The frequency spectrum of such a signal is not band limited, but we can identify a bandwidth W Hz, beyond which the signal energy is negligible.

Assuming a time limited signal that is also (approximately) band limited, the signal $x(t)$ with ideal sampling having N samples in $[0, T]$ interval can be expressed as

$$x(t) \approx \sum_{n=0}^{N-1} x(n\Delta t)\delta(t - n\Delta t)$$

where $\Delta t = \frac{T}{N}$ is the time between samples. Substituting for $x(t)$ in the Fourier transform formula, changing integration to summation and simplifying, we obtain

$$X(f) = \sum_{n=0}^{N-1} x(n\Delta t)e^{-j2\pi fn\Delta t}$$

Since we are interested in digital signals, f is restricted to the discrete set of values $\left[0, \frac{1}{T}, \frac{2}{T}, \frac{N-1}{T}\right]$, thus setting $f = \frac{k}{T} = \frac{k}{N\Delta t}$, the DFT becomes

$$X_k = \sum_{n=0}^{N-1} x_n e^{-j2\pi kn/N} \quad k = 0, 1, \dots, N - 1$$

and the *inverse Discrete Fourier Transform* IDFT becomes

$$x_n = \frac{1}{N} \sum_{k=0}^{N-1} X_k e^{j2\pi nk/N} \quad n = 0, 1, \dots, N - 1$$

The IDFT allows to recover x_n from its spectrum.

Discrete-Time Fourier Series (DTFS)

The Fourier series (FS) applies to the continuous-time periodic signals. Similarly, for discrete-time periodic signal we can obtain the discrete-time Fourier series (DTFS) of the signal. The trigonometric Fourier series coefficients for a CT signal were derived as follows:

$$a_0 = \frac{1}{T_0} \int_0^{T_0} x(t) dt$$
$$a_n = \frac{2}{T_0} \int_0^{T_0} x(t) \cos n\omega_0 t dt \quad n \neq 0$$
$$b_n = \frac{2}{T_0} \int_0^{T_0} x(t) \sin n\omega_0 t dt \quad n \neq 0$$

For a discrete periodic signal of period T with N samples in a period the trigonometric Fourier series coefficients similar to the CT signals can be obtained as

$$a_0 = \frac{2}{N} \sum_{k=0}^{N-1} x_k$$
$$a_n = \frac{2}{N} \sum_{k=0}^{N-1} x_k \cos \frac{2\pi n}{N} k \quad n \neq 0$$
$$b_n = \frac{2}{N} \sum_{k=0}^{N-1} x_k \sin \frac{2\pi n}{N} k \quad n \neq 0$$

Shannon's theorem applied to these functions states that if N samples are taken, then they only truly represent frequencies up to harmonic number $\frac{N}{2}$ (i.e., half the sampling frequency). If N is an odd number, then $\frac{N}{2}$ is not an integer. In this case, simply compute harmonics up to just below $\frac{N}{2}$, in other words, the highest possible harmonic would be $\frac{N-1}{2}$. The result is accurate for band limited signals (periodic signals). If the above coefficients are used for time limited signals such as a square pulse which are not bandlimited the result will not be accurate. However, if the sampling rate is increased the answer would be close but not quite accurate.

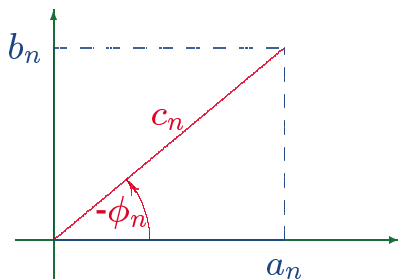
Once a_0 , a_n , and b_n are computed, they are used in the Fourier series expression to obtain $x(t)$

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\frac{N}{2}} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$$

An alternative expression is

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\frac{N}{2}} c_n \cos(n\omega_0 t + \phi_n)$$

where c_n and ϕ_n can be found from the following triangle



Example 10.1

A certain continuous-time function with the fundamental frequency of 200 Hz has the following values for 5 samples in the period.

$$\begin{array}{l} k : 0 \quad 1 \quad 2 \quad 3 \quad 4 \\ x_k : 3 \quad -5 \quad 2 \quad 7 \quad 0 \end{array}$$

Number of harmonics required is $n = \frac{N-1}{2} = \frac{5-1}{2} = 2$

$$a_0 = \frac{2}{N} \sum_{k=0}^4 x_k = \frac{2}{5} (3 - 5 + 2 + 7 + 0) = 2.8$$

The average value is $\frac{a_0}{2} = 1.4$

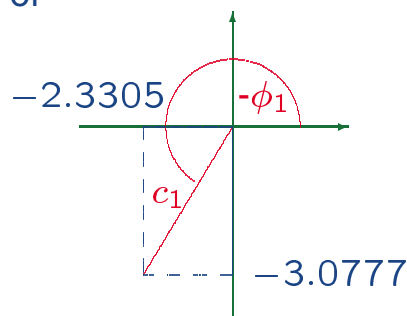
For 5 samples in a period $\frac{2\pi}{N} = \frac{(2)(360)}{5} = 72^\circ$, and we have

$$a_n = \frac{2}{5} \sum_{k=0}^4 x_k \cos(72n)k \quad \text{and} \quad b_n = \frac{2}{5} \sum_{k=0}^4 x_k \sin(72n)k$$

$$a_1 = \frac{2}{5} [3 \cos(0) - 5 \cos(72) + 2 \cos(72)2 + 7 \cos(72)3 + (0) \cos(72)4] = -2.3305$$

$$b_1 = \frac{2}{5} [3 \sin(0) - 5 \sin(72) + 2 \sin(72)2 + 7 \sin(72)3 + (0) \sin(72)4] = -3.0777$$

or



$$c_1 = [(2.3305)^2 + (3.0777)^2]^{\frac{1}{2}} = 3.86$$

$$-\phi_1 = 270 - \tan^{-1} \frac{2.3305}{3.0777}$$

$$= 232.86^\circ$$

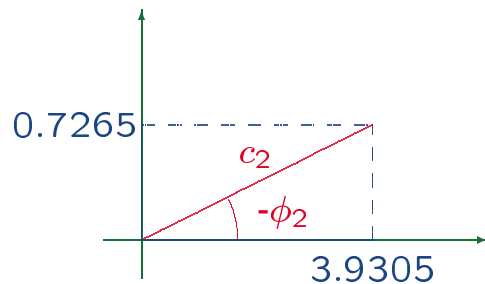
$$\phi_1 = 127.14^\circ$$

Similarly for the second harmonic, we have

$$a_2 = \frac{2}{5} [3 \cos(0) - 5 \cos(144) + 2 \cos(144)2 + 7 \cos(144)3 + (0) \cos(144)4] = 3.9305$$

$$b_2 = \frac{2}{5} [3 \sin(0) - 5 \sin(144) + 2 \sin(144)2 + 7 \sin(144)3 + (0) \sin(144)4] = 0.7265$$

or



$$c_2 = [(3.9305)^2 + (0.7265)^2]^{\frac{1}{2}} = 3.997$$

$$-\phi_2 = \tan^{-1} \frac{0.7265}{3.905}$$

$$= 10.54^\circ$$

$$\phi_2 = -10.54^\circ$$

The angular frequency $\omega_0 = 2\pi(200) = 400\pi$. Thus the expression for $x(t)$ is

$$x(t) = 1.4 + 3.86 \cos(400\pi t + 127.14^\circ) + 3.997 \cos(800\pi t - 10.5^\circ)$$

Fast Fourier Transform (FFT)

DFT applied to complex signals for large series takes considerable time (on the order of N^2) to compute. An efficient algorithm developed by Cooley and Tukey in 1965 known as fast Fourier Transform (FFT) reduces the number of computation considerably, The FFT is the most important discoveries in the field of digital signal processing. MATLAB Signal Processing Toolbox contain two functions **fft** and **ifft** that makes it very easy to obtain the DTFS and the inverse DTFS. For a vector x of length N representing one period of an N periodic signal x_n , we use the following command to obtain the DTFS coefficients

$$X = \text{fft}(x)/N$$

Similarly, the command

$$x = \text{ifft}(X)*N$$

returns a vector x that represents one period for the time signal when the spectrum X is defined.

MATLAB assume the summation from 0 to $N - 1$, so the first element of x and X corresponds to x_0 , and X_0 , respectively. Since **fft(x)** evaluates the sum of DTFS, division by N in **fft(x)** and multiplication by N in **ifft(X)** is necessary. To obtain the trigonometric DTFS coefficients a_n and b_n , **fft(x)** is divided by $N/2$, and **ifft(X)** is multiplied by $N/2$.

Example 10.2

Use The `fft` function to compute the DTFS coefficient of the sampled function described in Example 10.1.

We use the following commands:

```
N = 5; k= 0:1:N-1;
x_k = [3 -5 2 7 0 ];
C_n = fft(x_k)/(N/2);
disp([k', C_n'])
stem(200*k, abs(C_n))
xlabel('f'), ylabel('|C(f)|')
% We use ifft to reconstruct the signal
xrecon = ifft(C_n)*(N/2);
xrecon = xrecon.'
```

The result is

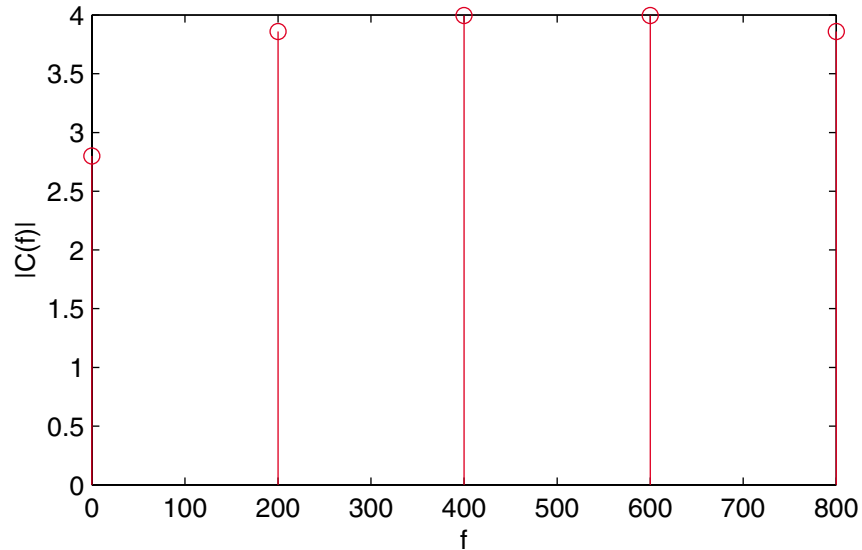
0	2.8000
1	-2.3305 - 3.0777i
2	3.9305 + 0.7265i
3	3.9305 - 0.7265i
4	-2.3305 + 3.0777i

```
xrecon =  
    3.0000 - 0.0000i  
   -5.0000 - 0.0000i  
    2.0000 - 0.0000i  
    7.0000 - 0.0000i  
    0.0000 + 0.0000i
```

The coefficients a_0 , a_1 , b_1 , a_2 , and b_2 are the same as the answers in Example 10.1.

Note that the reconstructed signal may have a negligibly small imaginary component which is an artifact of the numerical rounding errors.

The magnitude spectrum is as shown.



Note that to obtain the exponential DTFS coefficients X_n , $\text{fft}(x)$ is divided by N , and $\text{ifft}(X)$ is multiplied by N , instead of the factor $N/2$.

Example 10.3

Using **fft** Find the the DTFS coefficients for the signal

$$1 + 4 \cos\left(\frac{2\pi}{10}k + \frac{\pi}{3}\right)$$

Then use **ifft** to reconstruct the time-domain samples of the signal.

The signal period is 10, we obtain the DTFS coefficients for one period using the following commands.

```
N = 10; k= 0:1:N-1;
x = 1+ 4*cos(2*pi/10*k + pi/3);
X = fft(x)/N;
disp([k', X'])
subplot(2,1,1), stem(k/N, abs(X), 'r')
xlabel('f'), ylabel('|X(f)|')
%We use ifft to reconstruct the signal
xrecon = ifft(X)*N;
xrecon=real(xrecon)
subplot(2,1,2), plot(k/N,real(xrecon), 'o',k/N,x)
xlabel('f'), ylabel('x[n]')
legend('Samples from ifft','Original signal',0)
```

The result is

k	X[k]
0	1.0000
1	1.0000 - 1.7321i
2	-0.0000 - 0.0000i
3	-0.0000 - 0.0000i
4	-0.0000 - 0.0000i
5	0.0000 + 0.0000i
6	-0.0000 + 0.0000i
7	-0.0000 + 0.0000i
8	-0.0000 + 0.0000i
9	1.0000 + 1.7321i

```
xrecon = 3.0000  
         0.5819  
        -1.6765  
        -2.9126  
        -2.6542  
        -1.0000  
         1.4181  
         3.6765  
         4.9126  
         4.6542
```

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The magnitude spectrum and the reconstructed time-domain samples are as shown.

