# CHAPTER 3

## Sampled-Data Systems

A discrete signal $e(kT)$, or simply $e(k)$ obtained from an analog signal $e(t)$ is a number sequence that occurs repeatedly at instants of time $T$ seconds apart. T is the sampled period and $1/T$ is the sample rate in Hz. A discrete system is one whose operation is described by difference equation. A system having both discrete and continuous signals is called a sampled data system as shown in Figure 3.1.

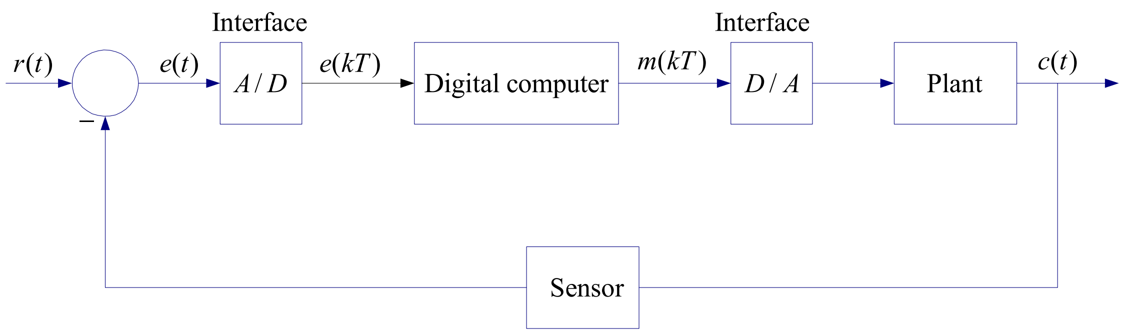


**Figure 3.1** Sampled data control system

### Digital-to-Analog Conversion

The D/A converter is a device that converts the sampled signal $m(kT)$ to a continuous signal $m(t)$. The weighted voltages are summed together to produce the analog output.

### Analog-to-digital Conversion

In the analog-to-digital converter, the analog signal is first converted to a sampled signal and then converted to a sequence of binary numbers. The sampling rate must be at least twice the bandwidth of the signal, or else there will be distortion. A satisfactory sampling rate is $20\omega_B$. The minimum sampling frequency is known as the *Nyquist sampling rate*. In Figure 3.2 the analog signal such as a voltage is sampled at periodic intervals and held over the sampling interval by a device called a *zero-order sample-and-hold* that yields a staircase approximation to the analog signal. Sampled are held in order to be digitalized by a digital counter.

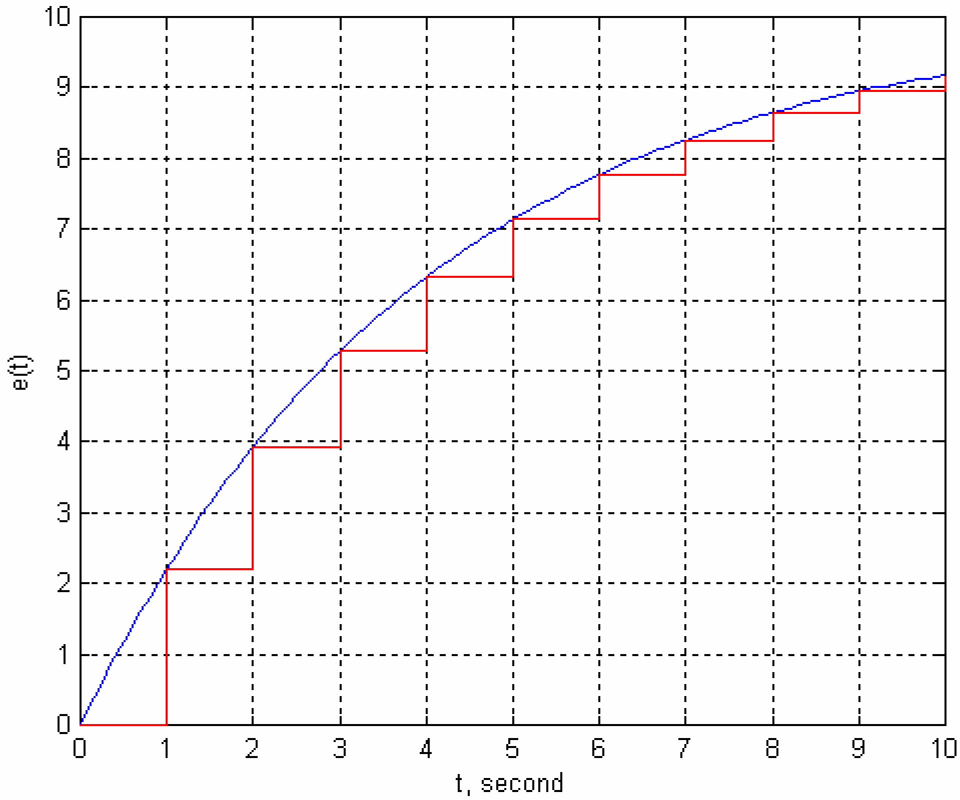


**Figure 3.1** Analog signal and the sample-and-hold signal

Table 3.1 gives a 4-bit coding of an analog signal that may range between 0 to 10 V. The information in the table is shown in graphical form in Figure 3.3.

| **Analog voltage** | **Binary Representation** |
|---|---|
| 0 - 2.212 | 0000 |
| 2.212 - 3.9347 | 0001 |
| 3.9347- 5.2763 | 0010 |
| 5.2763 - 6.3212 | 0011 |
| 6.3212 - 7.135 | 0100 |
| 7.135 - 7.7687 | 0101 |
| 7.7687 - 8.2623 | 0110 |
| 8.2623 - 8.6466 | 0111 |
| 8.6466 - 8.946 | 1000 |
| 8.946- 9.1792 | 1001 |
| 9.1792 - 10 | 1010 |

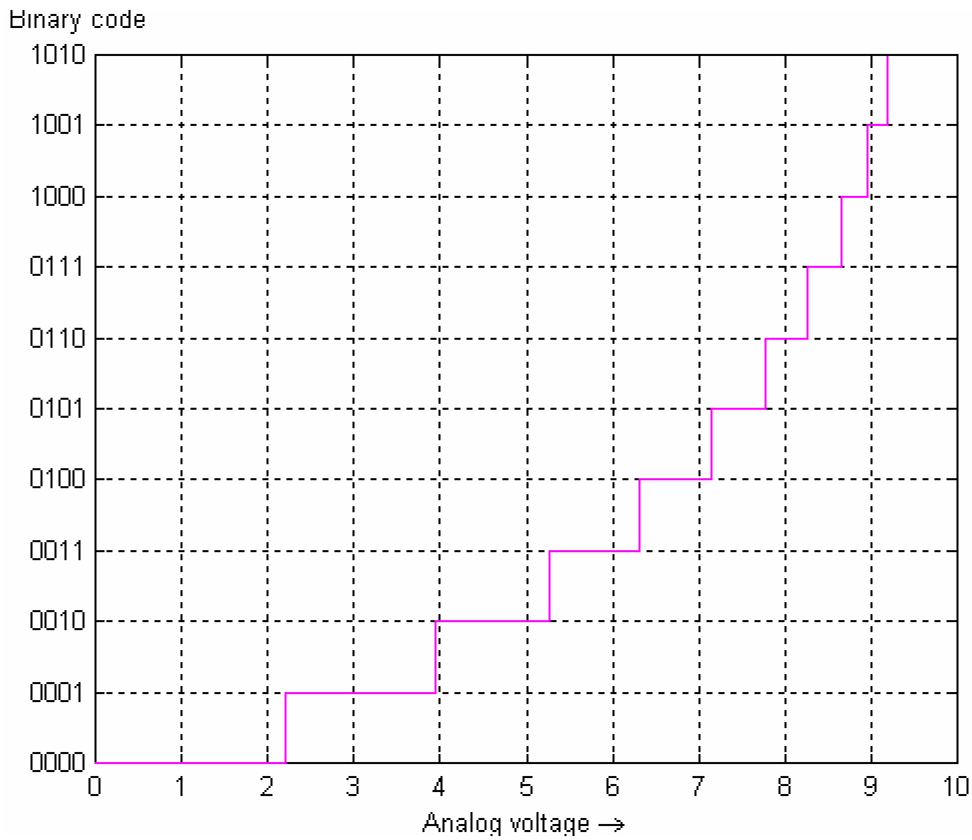


**Figure2.3** Conversion of samples to digital numbers.

Each binary number represents a range of analog voltage; hence there is a *quantization* error. For a 4-bit conversion, the maximum quantization error is $2^{-4} = 6.25\%$. The qunatization error in 16-bit conversion, for example, corresponds to a signal-to-noise ratio (SNR) of

$$\text{SNR (in dB)} = 20\log_{10} 2^{16} = 96.3\%$$

**Modeling the Digital Computer**

Since signals are sampled at specified intervals and held causes the system performance to change with changes in sampling rate. Thus the computer's effect upon the signal comes from this sampling and holding. In order to model the computer we must come up with a mathematical representation of this sample-and-hold process.

**Modeling the Sampler**

The objective is to derive a transfer function model for the digital computer as represented by a sampler and zero-order-hold. For the sampled signal the Laplace transform is replaced by a related transform known as the *z-transform*. Sampling provides the mechanism for converting analog signals to digital signals. Sampling of a continuous-time signal $e(t)$ can be illustrated by a rotating switch that is closed for an instant every $T$ seconds as shown in Figure 3.2.

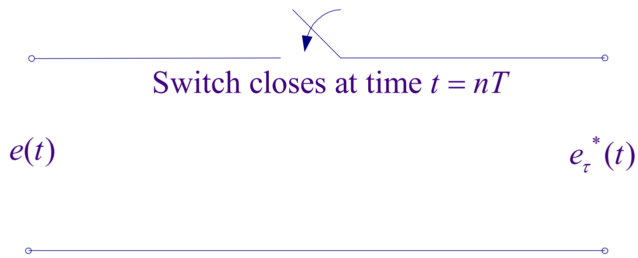


**Figure 3.4** Sampling illustrated by a switch

The action of the switch can be considered to be the product of $e(t)$ and a sampling function $p(t)$. Where $p(t)$ is a periodic pulse train of period $T$, and pulse duration $\tau$. The sampled signal $e^*(t)$ equals $e(t)$ for the $\tau$ seconds that the switch remains closed and is zero when the switch is open as shown in Figure 3.5.

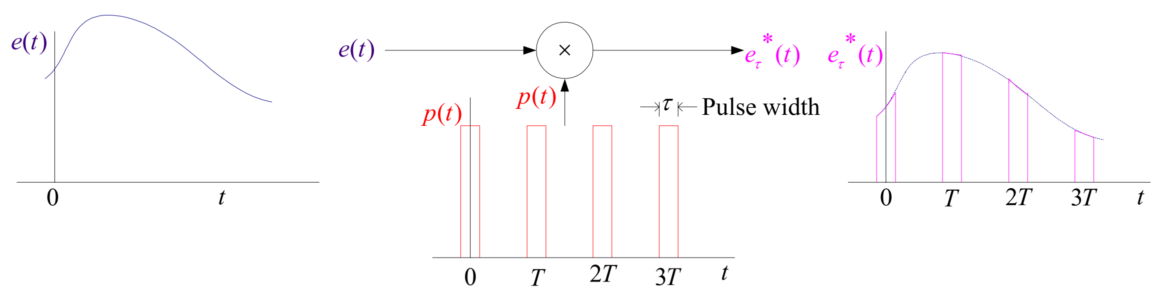


**Figure 3.5** A sampled waveform resulted by product of $e(t)$ with a pulse train.

The equation of the sampled wave from is

$$e_\tau^*(t) = e(t)p(t) = e(t)\sum_{k=-\infty}^{k=\infty} u(t-kT) - u(t-kT-\tau) \qquad k \text{ is an integer}$$

If the pulse width $\tau$ is small, $e(t)$ can be considered constant during the sampling period, and we have

$$e_\tau^*(t) = e(t)p(t) = \sum_{k=-\infty}^{k=\infty} e(kT)[u(t-kT) - u(t-kT-\tau)]$$

Taking the Laplace transform, we get

$$E_\tau^*(s) = \sum_{k=-\infty}^{k=\infty} e(kT)\left[\frac{e^{-kTs}}{s} - \frac{e^{-(kT+\tau)s}}{s}\right] = \sum_{k=-\infty}^{k=\infty} e(kT)\left[\frac{1-e^{-\tau s}}{s}\right]e^{-kTs}$$

Replacing $e^{-\tau s}$ by its series expansion, we have

$$E_\tau^*(s) = \sum_{k=-\infty}^{k=\infty} e(kT)\left[\frac{1-\left(1-\tau s+\dfrac{(\tau s)^2}{2!}-\cdots\right)}{s}\right]e^{-kTs}$$

For small $\tau$ we can neglect the higher terms,

$$E_\tau^*(s) = \sum_{k=-\infty}^{k=\infty} e(kT)\left[\frac{\tau s}{s}\right]e^{-kTs} = \sum_{k=-\infty}^{k=\infty} e(kT)\tau e^{-kTs} \tag{3.1}$$

Transforming back to time-domain, we have

$$e_\tau^*(t) = \tau \sum_{k=-\infty}^{k=\infty} e(kT)\delta(t-kT) \tag{3.2}$$

Thus, the result of sampling with rectangular pulses can be considered as a series of impulse functions of strength $\tau f(kT)$. The above equation can be considered as ideal sampler cascaded with $\tau$, as shown in Figure 3.6.

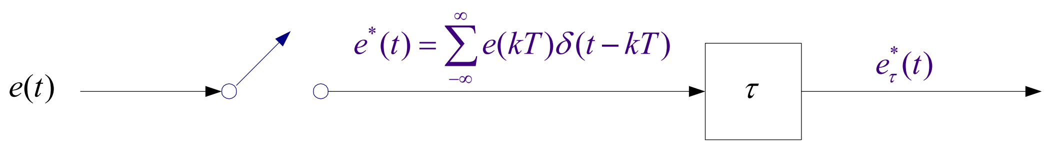


**Figure 3.6** Model of sampling with a rectangular pulse train.

The ideal sampler for $k \geq 0$ is

$$E^*(s) = \sum_{k=0}^{\infty} e(kT)e^{-kTs} \tag{3.3}$$

$$e^*(t) = \sum_{k=0}^{\infty} e(kT)\delta(t-kT) \tag{3.4}$$

Note the similarity between the starred transform, $E^*(s)$, and the z-transform $E(z)$. The starred transform is defined as

$$E^*(s) = e(0) + e(T)e^{-Ts} + e(2T)e^{-2Ts} + \cdots$$

and the z-transform is defined as

$$E(z) = e(0) + e(1)z^{-1} + e(2)z^{-2} + \cdots$$

**Zero-Order Hold**

A sample-and-hold device is used to hold the previously converted signal while a new conversion takes place as shown in Figure 3.6. If we assume an ideal sampler (i.e. $\tau = 1$) then $e^*(t)$ is represented by a sequence of delta functions. The result is an output signal that changes in a staircase approximation to $e(t)$. Hence the output from the hold $e_h(t)$ is a sequence of step functions whose amplitude is $e(kT)$ at sampling instant. For an impulse at time $t = 0$, the Laplace transform of its resulting step, which starts at $t = 0$, and end at $t = T$ is

$$G_h(s) = \frac{1 - e^{-Ts}}{s} \tag{3.5}$$

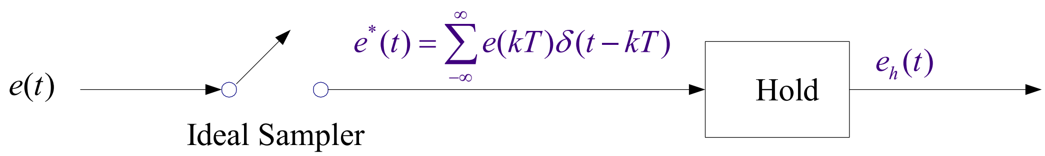


**Figure 3.7** Ideal sampler and zero-order hold.

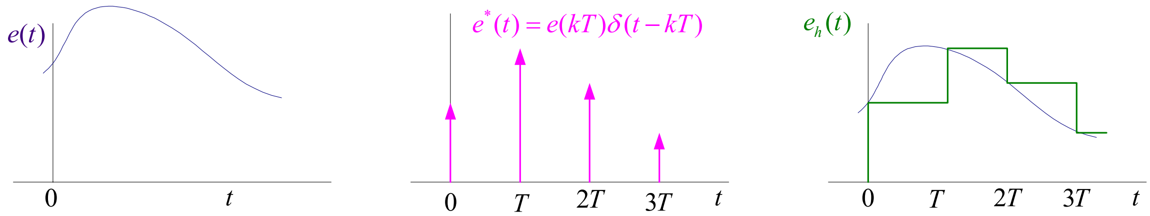


**Figure 3.8** Ideal sampling and the zero-order hold.

From 2.3 the s-domain sample-hold operation is represented in Figure 3.9.

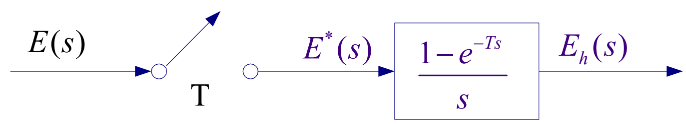


**Figure 3.9** Representation of sampler/data hold

Where the starred transform $E^*(s)$, is given by 2.3. Therefore $E_h(s)$ also shown as $\bar{E}(s)$ is

$$E_h(s) = \frac{1-e^{-Ts}}{s}\sum_{k=0}^{\infty} e(kT)e^{-kTs} \tag{3.6}$$

Recall that in the previous chapter the complex variable $z$ was defined by

$$z = e^{sT} \tag{3.7}$$

**Example 3.1**

Find $E^*(s)$ if the input is a unit step, i.e., $e(kT) = 1$, for $k = 1,\ 2,\ \cdots$.

$$E^*(s) = \sum_{k=0}^{\infty} e(kT)e^{-kTs} = e(0) + e(T)e^{-Ts} + e(2T)e^{-2Ts} + \cdots$$

$$= 1 + e^{-Ts} + e^{-2Ts} + \cdots$$

or

$$E^*(s) = \frac{1}{1-e^{-Ts}}$$

Substituting from 2.7

$$E^*(s) = \frac{1}{1-z^{-1}} = \frac{z}{z-1}$$

The above transfer function is the z-transform of the unit step and we see the similarity between the starred transform and the z-transform, i.e.,

$$E^*(s) = E(z)\big|_{z=e^{sT}} \tag{3.5}$$

**Pulse Transfer Function**

We now apply the z-transform to obtain the transfer function of a sampled-data system. Consider the system shown in Figure 3.10(a).

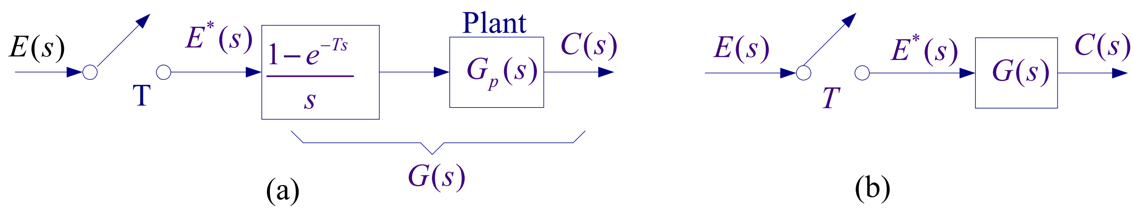


**Figure 3.10** Open-loop sampled-data system.

We denote the product of the zero-order hold transfer function and transfer function as $G(s)$ as shown in Figure 3.10(b), i.e.,

$$G(s) = \frac{1-e^{-sT}}{s} G_p(s) \tag{3.6}$$

From now on the sampled-data transfer function is understood to contain the zero-order hold, so

$$C(s) = G(s)E^*(s) \tag{3.7}$$

If the input is sampled, the output is still continuous. If, however, we are satisfied with finding the output at the sampling instant and not in between, we can add a phantom sampler, which is in synchronism with the input as shown in Figure 3.11.

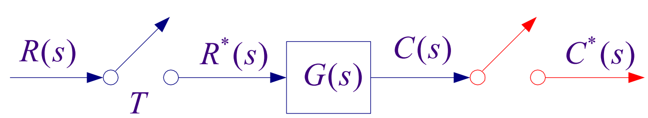


**Figure 3.11** Open-loop sampled-data with phantom sampler.

$$r^*(t) = \sum_{n=0}^{\infty} r(nT)\delta(t-nT) \tag{3.8}$$

Since the impulse response of a transfer function $G(s)$, is $g(t)$, we can write the time output of $G(s)$ as the sum of impulse responses generated by the input, thus

$$c(kT) = \sum_{n=0}^{\infty} r(nT)g(k-n)T \tag{3.9}$$

$$C(z) = \sum_{k=0}^{\infty} c(kT)z^{-k} \tag{3.10}$$

Substituting (3.9) into (3.10), we get

$$C(z)=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} r(nT)g[(k-n)T]z^{-k}$$

Letting $m=k-n$

$$C(z)=\sum_{m+n=0}^{\infty}\sum_{n=0}^{\infty} r(nT)g(mT)z^{-m-n}$$

Since $m+n=0$, or $m=-n$ yield negative values of $m$ for $n>0$, can change the lower limit from $m+n=0$ to $m=0$. The above expression is written as

$$C(z)=\sum_{m=0}^{\infty} g(mT)z^{-k}\sum_{n=0}^{\infty} r(nT)z^{-n} \tag{3.11}$$

or

$$C(z)=G(z)R(z) \tag{3.12}$$

Equation (3.12) is a very important result, since it shows that the transform of the sampled output is the product of the transforms of the sampled input and the pulsed transfer function of the system. Recall that although the output of the system is continuous function, we had to make an assumption of a sampled output (phantom sampler) in order to arrive at the compact result of Equation (3.12).

**Converting $\mathbf{G_p(s)}$ plant transfer function in cascade with z.o.h. to sampled-data transfer function G(s)**

One way of finding the pulse transfer function $G(z)$, is to start with $G(s)$, find $g(t)$, and then use the z-transform table to find $G(z)$. This is illustrated in the following examples.

**Example 3.2 (Example 3.6 textbook)**

Given a z.o.h. in cascade with $G_p(s)=\dfrac{1}{s+1}$ or

$$G(s)=\frac{1-e^{-Ts}}{s}\frac{1}{s+1}$$

(a) Find the sampled data transfer function $G(z)$

(b) Evaluate $G(z)$ if the sampling time $T=1$ second

We can find a general solution by moving the s in the denominator of the z.o.h. to $G_p(s)$, i.e.,

$$G(s)=(1-e^{-Ts})\frac{G_p(s)}{s}$$

from which

$$G(z)=(1-z^{-1})\mathcal{Z}\left[\frac{G_p(s)}{s}\right]=(\frac{z-1}{z})\mathcal{Z}\left[\frac{G_p(s)}{s}\right] \tag{3.13}$$

We now find the inverse Laplace transform of $G_2(s) = \dfrac{G_p(s)}{s}$

$$G_2(s) = \frac{G_p(s)}{s} = \frac{1}{s(s+1)} = \frac{1}{s} + \frac{-1}{s+1}$$

Therefore,

$$g_2(t) = 1 - e^{-t} \quad \Rightarrow \quad g_2(kT) = 1 - e^{-kT}$$

From the z-transform table

$$G_2(z) = \frac{z}{z-1} - \frac{z}{1-e^{-T}}$$

Substituting in (3.13)

$$G(z) = \frac{z-1}{z}\left[\frac{z}{z-1} - \frac{z}{z-e^{-T}}\right] = 1 - \frac{z-1}{z-e^{-T}} = \frac{1-e^{-T}}{z-e^{-T}}$$

(b) For $T = 1$ second

$$G(z) = \frac{1-e^{-1}}{z-e^{-1}} = \frac{0.6321}{z-0.3679}$$

MATLAB control system toolbox function **Gz = c2d(Gs, T, 'method')** converts the continuous-time LTI model Gs to a discrete-time model Gz with sample time T. The string **'method'** selects the discretization method among the following:

| | |
|---|---|
| **'zoh'** | Zero-order hold on the inputs |
| **'foh'** | Linear interpolation of inputs (triangle appx.) |
| **'tustin'** | Bilinear (Tustin) approximation |
| **'prewarp'** | Tustin approximation with frequency prewarping. |

The default is 'zoh' when METHOD is omitted. Also function **c2dm** can be used when either num and den of a transfer function or the A, B, C, D parameters of the state models are defined. That is we use **[Ad, Bd, Cd, Dd] = c2dm(A, B, C, D,T , 'method')** which converts the continuous-time state-space system (A, B, C, D) to discrete time using 'method': Also **[numd, dend] = c2dm(num, den ,T, 'method')** converts the continuous-time polynomial transfer function to discrete time using 'method'.

To find the sampled-data transfer function for $G(s)$ in Example 3.2, we use the following commands:

```
num = [0  1];
den = [1  1];
Gs = tf(num, den)
T = 1                    % sampling time
Gz = c2d(Gs, T, 'zho')
```

The result is

Transfer function:

```
 0.6321
----------
z - 0.3679
```

Sampling time: 1

This is the same result as found in Example 3.2.

**Example 3.3 (Example 12.7 Textbook)**

Find the step response of the sampled-data system in Example 3.2 when sampling time T=1 second.

C(z) = G(z)E(z)

For step input $E(z) = \frac{z}{z-1}$ and from Example 3.2 $G(z) = \frac{1-e^{-1}}{z-e^{-1}} = \frac{0.6321}{z-0.3679}$

Therefore

$$C(z) = \frac{0.632}{z-0.3679}\frac{z}{z-1}$$

or

$$\frac{C(z)}{z} = \frac{0.632}{z-0.3679}\frac{1}{z-1} = \frac{1}{z-1} + \frac{-1}{z-0.3679}$$

Therefore

$$C(z) = \frac{z}{z-1} + \frac{-z}{z-0.3679}$$

The inverse z-transform is

$$C(Kt) = 1-(0.3679)^K$$

For $k = 0,1,2,\cdots$

$$c(0) = 1-(0.3679)^0 = 0$$

$$c(1) = 1-(0.3679)^1 = 0.6321$$

$$c(2) = 1-(0.3679)^2 = 0.8646$$

$$c(3) = 1-(0.3679)^3 = 0.9502$$

etc.

We can use MATLAB to find the solution of Example 3.2-2.3. (chd3ex3.m)

We use the following commands:

```
numg = [0  1];
deng = [1  1];
T = 1;
[numz, denz] = c2dm(numg, deng, T, 'zho') % discrete-time num, den
dstep(numz, denz), grid
```

The result is

```
numz =
        0    0.6321
denz =
        1.0000   -0.3679
```

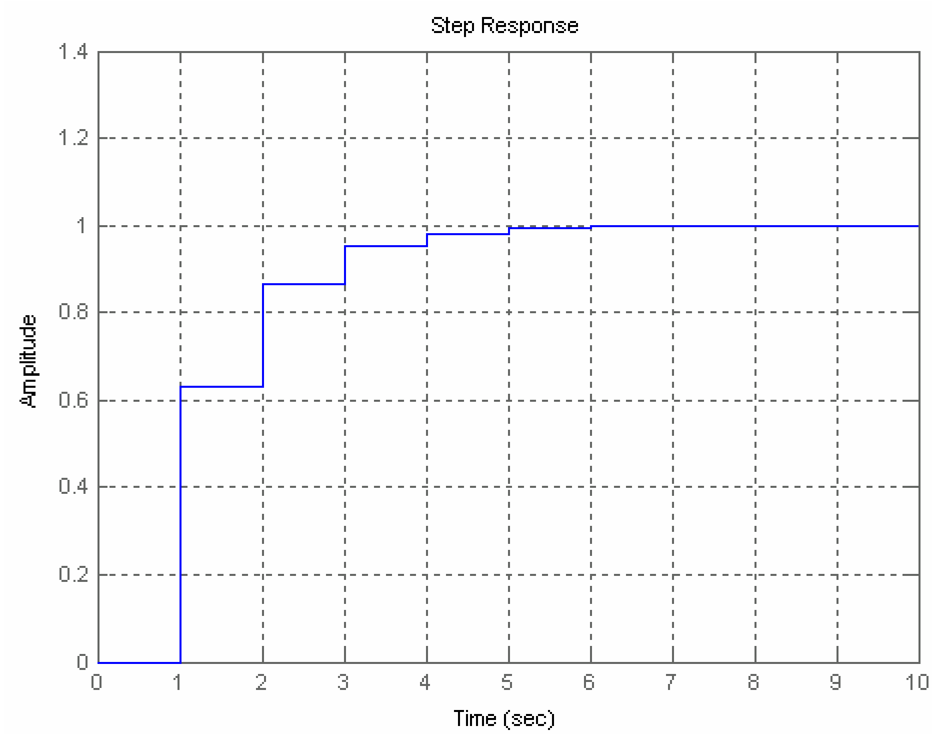

**Figure 3.12** Response for Example 3.3

In Simulink we can use the discrete transfer function to simulate the response in Example 3.3. (SimChd3ex3.mdl)

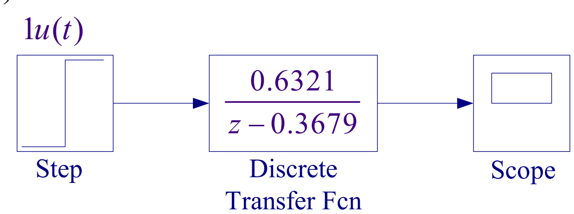

**Figure 3.13** Simulink model for Example 3.3.

The result is as shown.

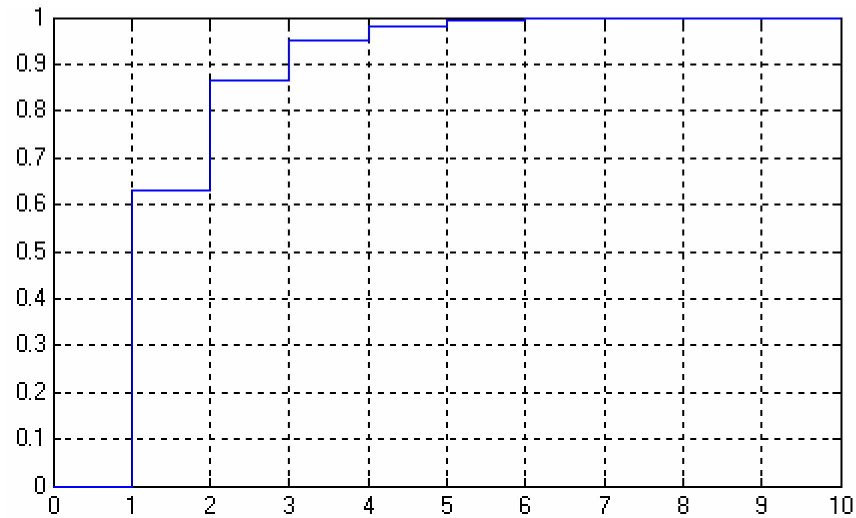


**Figure 3.14** Simulink response for Example 3.3.

The Scope plot in SIMULINK 4.0 is in yellow color. To produce a Figure plot and change the trace color I have written a script function named **plotscope**. To produce a figure plot type plotscope at MATLAB prompt, then click on the scope (outside the plot area) and hit Enter. The plot in Figure 3.14 is produced.

**Block Diagram Reduction**

When manipulating block diagrams for sampled-data systems, you must be careful and remember the definition of the sampled data system transfer function. For example, $\mathcal{Z}\{G_1(s)G_2(s)\} \neq G_1(z)G_2(z)$. That is, $G_1(s)$, $G_2(s)$ must be multiplied together before taking the z-transform. We use the notation $\mathcal{Z}\{\overline{G_1G_2}(s)\} = \overline{G_1G_2}(z)$.

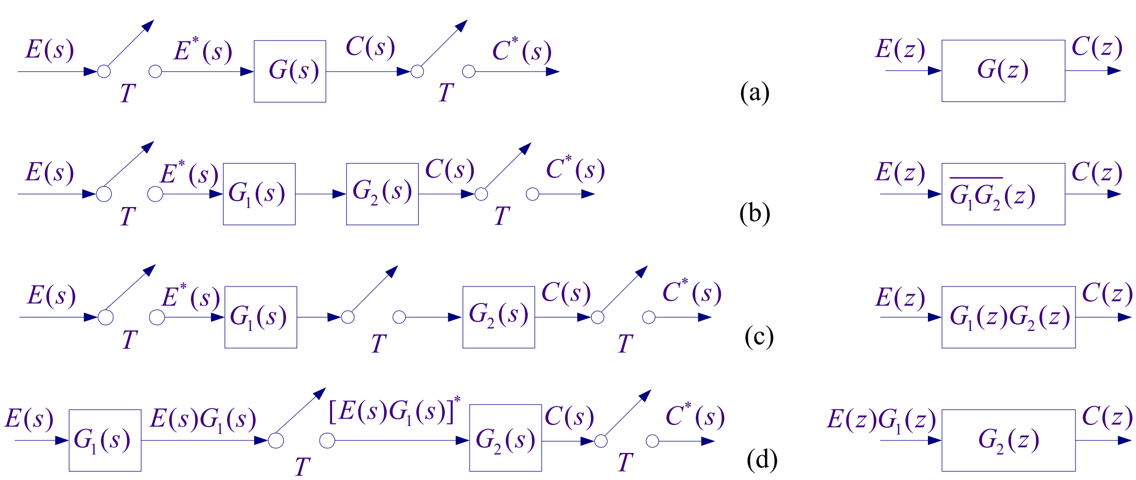


**Figure 3.15** Sampled-data systems and their z-transform

One operation we can always perform is to place a phantom sampler at the output of any subsystem that has a sampled input, provided that the nature of the signal sent to any other subsystem is not changed. Figure 3.15(a) is the standard pulse transfer function.

Figure 3.15(b) there is no sampler between $G_1(s)$ and $G_2(s)$, thus we have a single transfer function denoted by $\overline{G_1G_2}(s)$, Thus $C(z) = \overline{G_1G_2}(z)E(z)$. In Figure 3.15(c) the output is $C(z) = G_1(z)G_2(z)E(z)$. Similarly the output in Figure 3.15(d) is $C(z) = G_1(z)G_2(z)E(z)$.

**Example 3.4 (Example 12.8 textbook)**

The sampled-data open-loop control system contains a digital filter or digital controller as shown in Figure 12.16. Find the step response of the system.

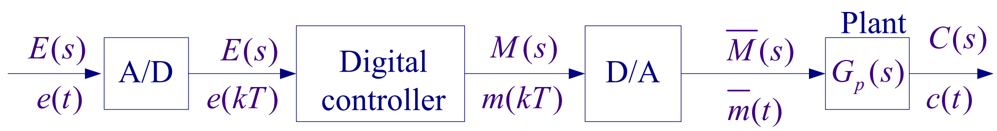


**Figure 12.16** Open-loop system with digital controller

The continuous-time signal $e(t)$ is converted into the number sequence $e(kT)$. The digital controller processes this number and produces the output number sequence $m(kT)$. The D/A converts the umber sequence $m(kT)$ into a continuous signal which will act on the plant dynamic. Representing the digital controller by the transfer function $D(z)$, the open-loop sampled data block diagram is obtained as shown in Figure 12.17.

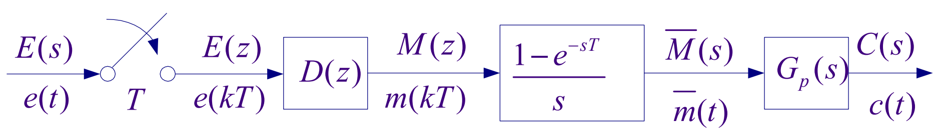


**Figure 12.17** Sampled-data control system.

$$M(z) = D(z)E(z)$$

$$C(s) = G_p(s)\overline{M}(s) = G_p(s)\left[\frac{1-e^{-Ts}}{s}\right]M(z)$$

Therefore,

$$C(z) = \mathcal{Z}\{G_p(s)\overline{M}(s)\} = \mathcal{Z}\left\{\frac{1-e^{-Ts}}{s}G_p(s)\right\}M(z)$$

$$C(z) == \mathcal{Z}\left\{\frac{1-e^{-Ts}}{s}G_p(s)\right\}D(z)E(z) = G(z)D(z)E(z)$$

Let $G_p(s) = \dfrac{1}{s+1}$ and suppose the controller is a PD controller defined by the difference equation

$$m(kT) = 2e(kT) - e(k-1)T$$

Finding the z-transform of the above equation

$$M(z) = 2E(z) - z^{-1}E(z) \quad \Rightarrow \quad D(z) = \frac{M(z)}{E(z)} = 2 - z^{-1} = \frac{2z-1}{z}$$

$$G(z) == \mathcal{Z}\left\{\frac{1-e^{-Ts}}{s}G_p(s)\right\} = (1-e^{-Ts})\mathcal{Z}\left\{\frac{G_p(s)}{s}\right\}$$

$$\frac{G_p(s)}{s} = \frac{1}{s(s+1)} = \frac{1}{s} + \frac{-1}{s+1} \quad \Rightarrow \quad g_2(t) = 1 - e^{-t}$$

$$G(z) = (1-z^{-1})\mathcal{Z}\left\{1-e^{-kT}\right\} = \frac{z-1}{z}\left[\frac{z}{z-1} - \frac{z}{z-e^{-T}}\right] = 1 - \frac{z-1}{z-e^{-T}} = \frac{1-e^{-T}}{z-e^{-T}}$$

$$C(z) = G(z)D(z)E(z) = \frac{1-e^{-T}}{z-e^{-T}}\frac{2Z-1}{z}\frac{z}{z-1} = \frac{(2z-1)(1-e^{-T})}{(z-1)(z-e^{-T})}$$

Let $C(z) = z^{-1}F(z)$

Then

$$\frac{F(z)}{z} = \frac{(2z-1)(1-e^{-T})}{(z-1)(z-e^{-T})} = \frac{1}{z-1} + \frac{1-2e^{-T}}{z-e^{-T}}$$

or

$$F(z) = \frac{z}{z-1} + \frac{(1-2e^{-T})z}{z-e^{-T}} \quad \Rightarrow \quad f(k) = 1 + (1-2e^{-T})e^{-kT}$$

Since $C(z) = z^{-1}F(z)$, then

$$c(k) = [1 + (1-2e^{-T})e^{-(k-1)T}u(k-1)$$

**Closed-Loop Sampled-Data Transfer Function**

To obtain the closed-loop sampled-data transfer function, the following phantom samplers are added to the block diagram in Figure 3.18(a). Since the response in (a) is valid only the sampling instants a sampler $S_4$ can be added. Also $S_2$ and $S_3$ are added at the inputs of the summing point whose output is sampled, which results in the block diagram shown in Figure 3.18(b). Next we move the sampler $S_1$ and the block $G_1(s)$ ahead of the pickoff point.

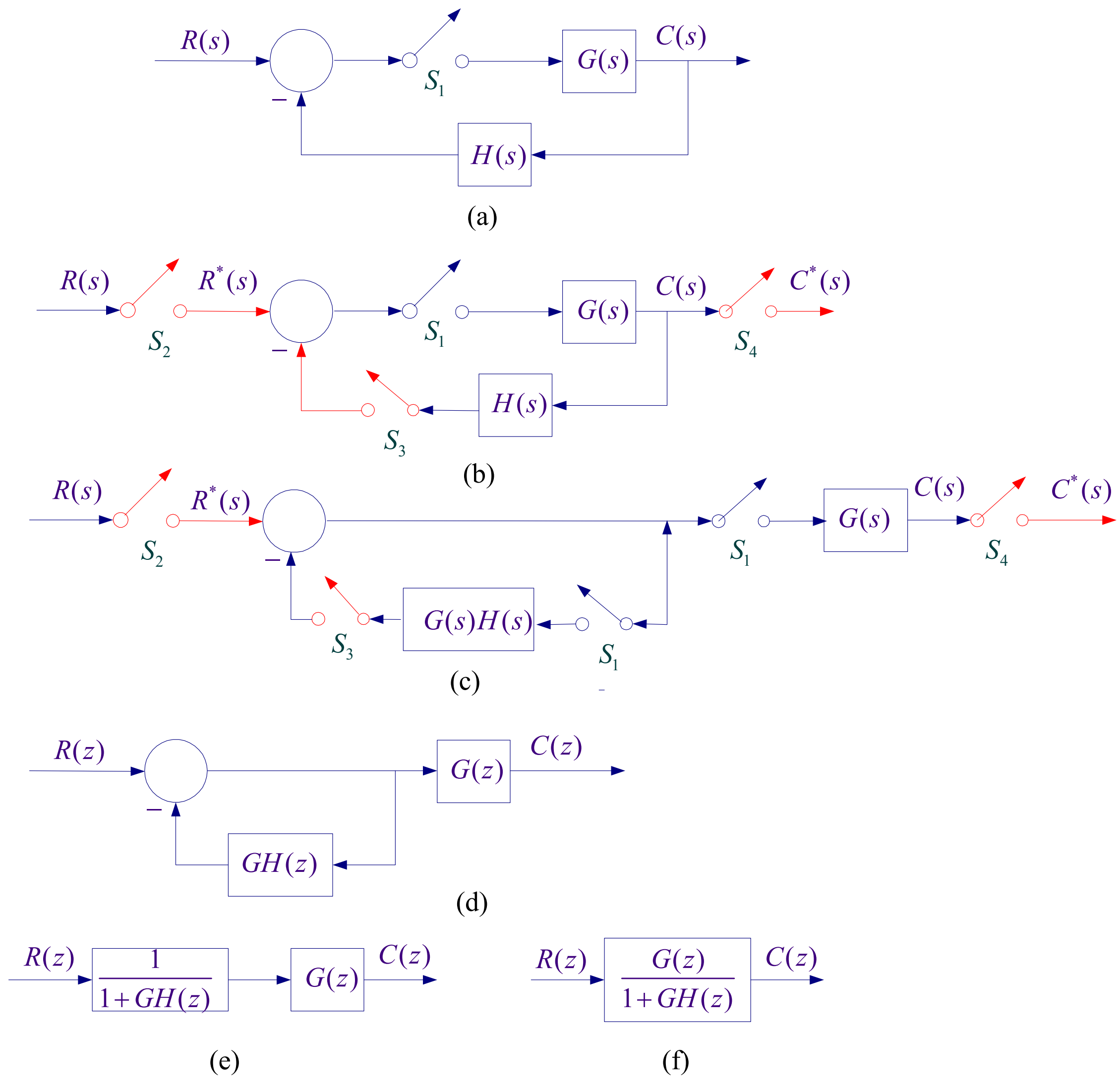


**Figure 3.18** Steps in block diagram reduction of a sampled-data system.

The result is that the feedback path would be multiplied by $G(s)$ as shown in Figure 3.18(c). The closed-loop system has now a sampled input and a sampled output. Since each block has a sampler we can represent the system in z-domain as shown in Figure 3.18(d). Reducing the z-domain block diagram yields the final result shown in Figure 3.18(f).