CHAPTER 5

Phase-Lag Design using the Bode Diagram

The phase-lag controller transfer function is

$$G_c(s) = \frac{K_c(s+\omega_0)}{(s+\omega_p)}\text{, where } \omega_p < \omega_0$$

The frequency response transfer function is

$$G_c(j\omega) = \frac{K_c(\omega_0 + j\omega)}{(\omega_p + j\omega)}$$

The transfer function of the phase-lag controller written in Bode diagram form is

$$G_c(j\omega) = \frac{K_c\omega_0}{\omega_p}\left(\frac{1+\frac{j\omega}{\omega_0}}{1+\frac{j\omega}{\omega_p}}\right)$$

Assuming a dc gain of unity for the controller, (i.e., $\frac{K_c\omega_0}{\omega_p} = 1$) the transfer function is

$$G_c(j\omega) = \frac{1+\frac{j\omega}{\omega_0}}{1+\frac{j\omega}{\omega_p}} \tag{5.1}$$

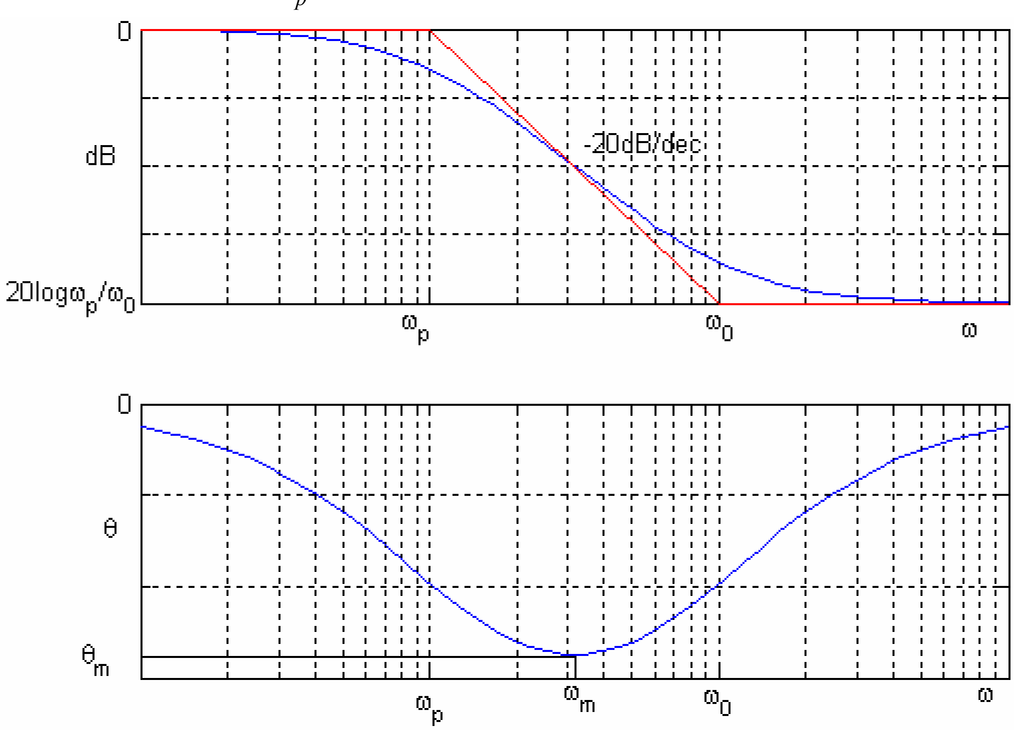

Figure 5.1 Bide diagram for phase-lag controller

The purpose of the controller is to reduce the steady-state error and to maintain a desired phase margin for a satisfactory transient response. The pole and zero of the phase-lag controller must be located substantially lower than the new gain crossover frequency. Therefore we must move the new gain crossover frequency to a lower frequency resulting in the desired phase margin while keeping the phase curve of the Bode plot relatively unchanged at the new gain crossover frequency.

The procedure for the phase-lag controller design is as follows:

1. Set the gain K to the value that satisfies the steady-state error specification and plot the Bode diagram
2. Typically the phase-lag controller contributes a small phase lag $5^\circ - 10^\circ$ phase lag. On the phase angle plot locate the frequency to yield the desired phase margin plus say 6°. This is the new gain crossover frequency $\omega_{gc}^{'}$
3. Determine the magnitude at $\omega_{gc}^{'}$ frequency. This is the required attenuation of the phase-lag controller such that the resultant magnitude plot would go through the 0 dB at this frequency. Draw the straight line high-frequency asymptote at this attenuation $dB(\omega_0)$.
4. Select the controller zero (break frequency ω_0) to be one decade below $\omega_{gc}^{'}$. That is $\omega_0 = 0.1\omega_{gc}^{'}$
5. At the high frequency asymptote from $dB(\omega_0) = 20\log\dfrac{\omega_0}{\omega_p} = 20\log\alpha$ find α.

   Determine the controller pole (break frequency ω_p), $\omega_p = \dfrac{\omega_0}{\alpha}$
6. Draw the compensated Bode plot.

Example 5.1

Consider a control system with the following open-loop transfer function

$$GH(s) = \frac{8}{s(s+1)(s+4)} = \frac{8}{s^2+5s+4}$$

Design a phase lag-controller such that the compensated system would result in the following specifications

1. Steady-state error of 0.05 due to a ramp input
2. Frequency response phase margin of 45°

$$e_{ss} = \frac{1}{K_v} = 0.05 \quad \Rightarrow \quad K_v = 20$$

The required gain is given by

$$K_v = \lim_{s\to 0} sG_c(s) = \frac{8K}{4} = 20 \quad \Rightarrow \quad K = 10$$

The Bode plot for $\frac{8K}{s^2+5s+4}$ with $K = 10$ to satisfy the error specification is obtained as shown in Figure 5.2. As it can be seen with this gain $\omega_{gc} > \omega_{pc}$, system is unstable with $GM = -12dB$ and $\phi_m = -28^\circ$.

We add 6° to the specified phase margin so the required phase margin is $\phi_m' = 45^\circ + 6^\circ = 51^\circ$. Therefore, we locate the frequency ω_{gc}' where the phase is $-180^\circ + 51^\circ = -129^\circ$. This is found to be $\omega_{gc}' = 0.598$

From the magnitude curve, the required attenuation of the controller is found to be 29 dB.

The zero of the controller is selected one decade below this crossover frequency, i.e., $\omega_0 = 0.1\omega_{gc}' = 0.1(0.598) \simeq 0.06$

From step 5, $dB(\omega_0) = 20\log\alpha = 29 \quad \Rightarrow \quad \alpha = 28$

The controller pole is given by

$$\omega_p = \frac{\omega_0}{\alpha} = \frac{0.06}{28} = 0.0021$$

Therefore the controller transfer function is

$$G_c(s) = \frac{0.035(s+0.06)}{(s+0.0021)}$$

Base on a dc gain of unity $\frac{K_c\omega_0}{\omega_p} = 1 \Rightarrow K_c = \frac{\omega_p}{\omega_0} = \frac{0.0021}{0.06} = 0.035$

The compensated open-loop transfer function is

$$KGH(s)G_c(s) = \frac{(8)(10)(0.035)(s+0.06)}{s(s+0.0021)(s^2+5s+4)} = \frac{2.8(s+0.06)}{s(s+0.0021)(s^2+5s+4)}$$

The velocity error constant is $K_v = \frac{2.8(0.06)}{(0.0021)(4)} = 20 \quad \Rightarrow \quad e_{ss} = \frac{1}{20} = 0.05$

Also as it can be seen from Figure 5.3 the phase margin is 45°.

If the gain $K = 10$ is included in the controller instead of the in the plant transfer function, the controller transfer function becomes

$$G_c(s) = \frac{0.35(s+0.06)}{(s+0.0021)}$$

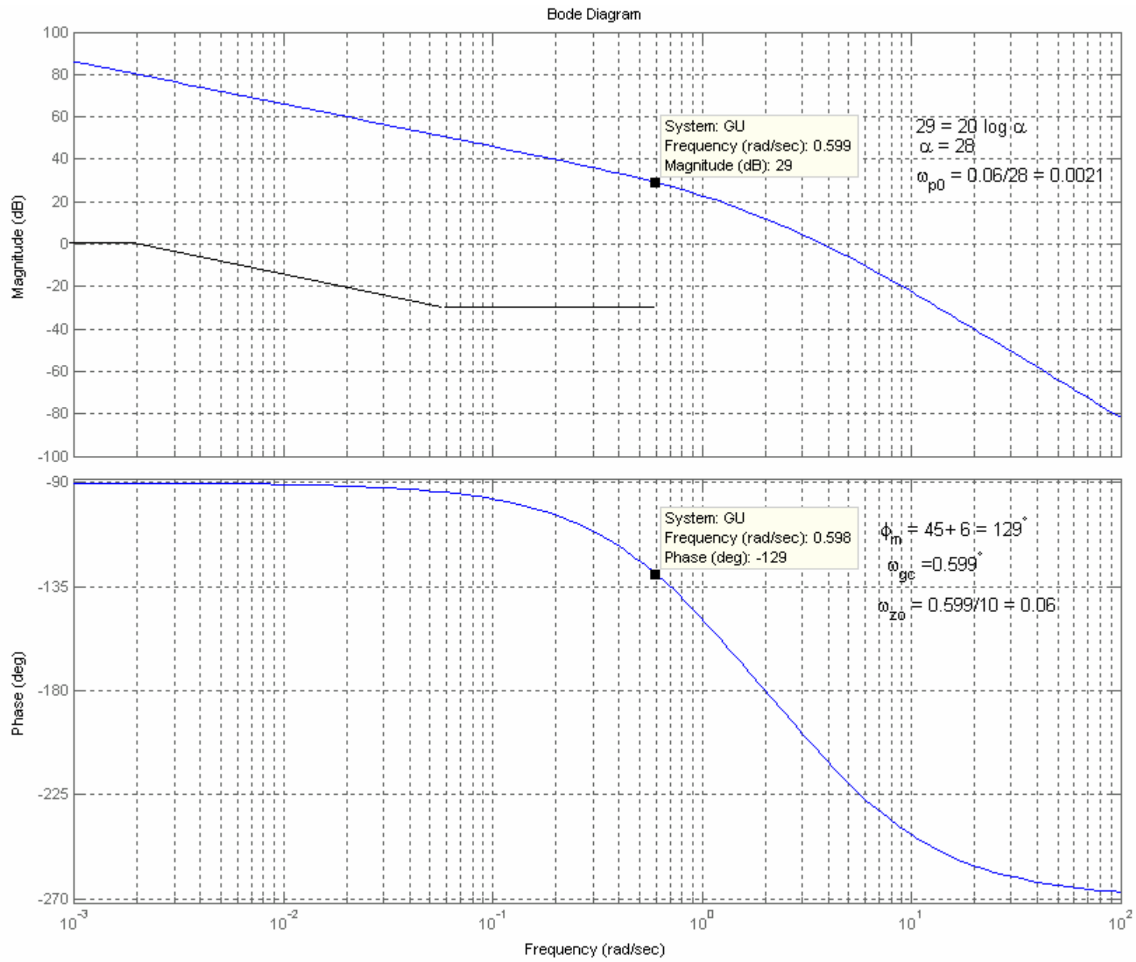

Figure 5.2 Bode diagram for Example 5.1, $K = 10$ to satisfy the steady-state error

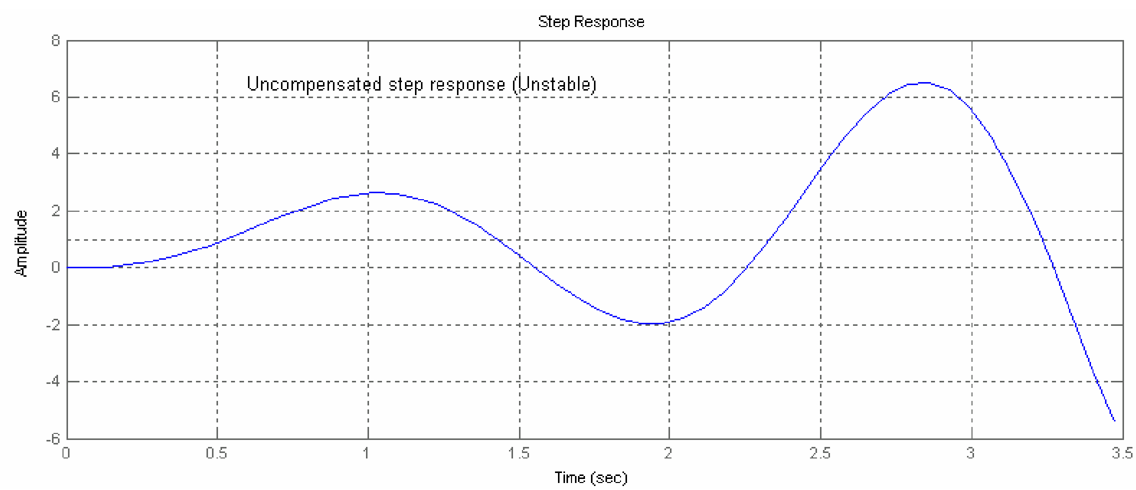

Figure 5.3 Uncompensated step response

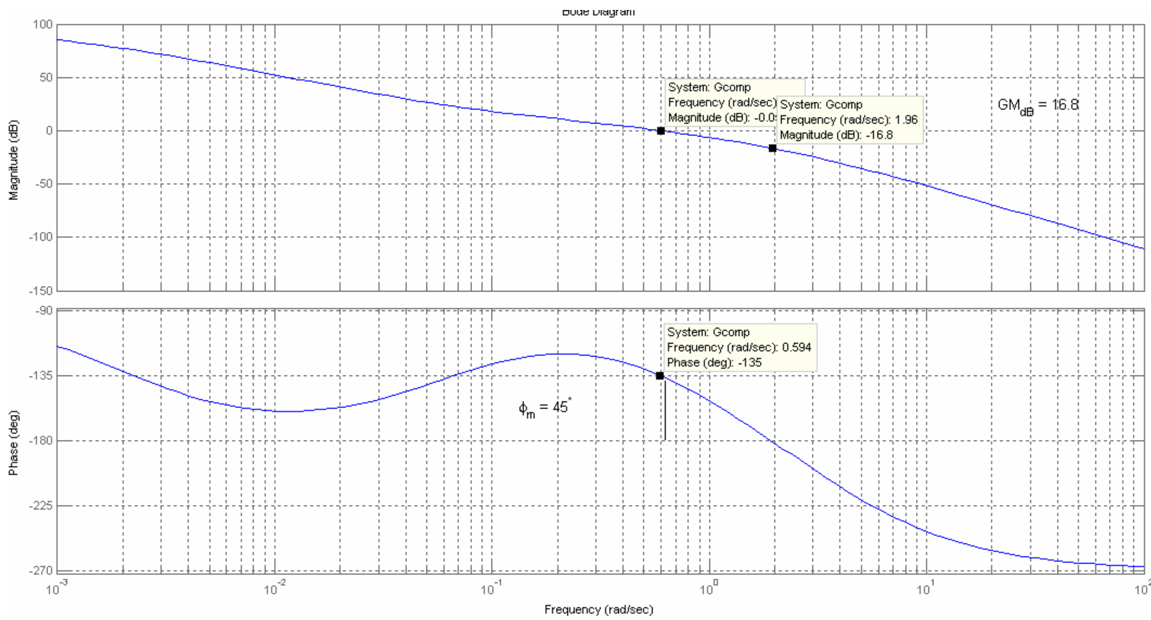

Figure 5.4 The compensated Bode diagram

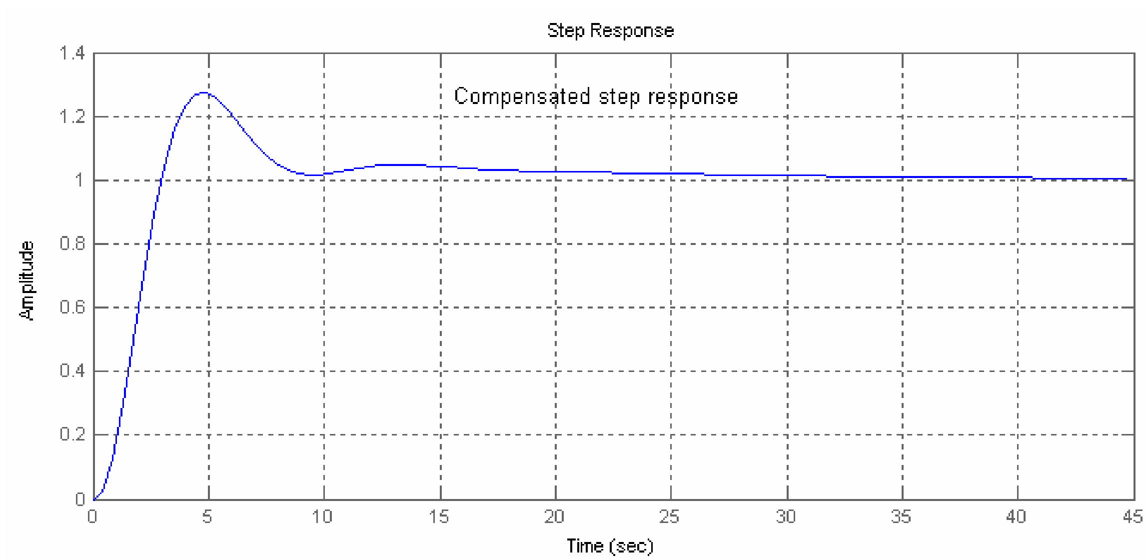

Figure 5.5 Compensated step response

The following MATLAB commands are used in this Example

```
Kvu=2; Kvc = 20;  K =10;
num=8;
den=[1  5  4  0];
disp('Unompensated open-loop transfer function')
GU = tf(num*K, den)              % Open-loop transfer function with K =10
w=logspace(-3,2, 100);           % specified range for w 10^-3 to 10^2
bode(GU, w); grid on             % Returns the Bode magnitude and phase plot
disp('Uncompensated GM, PM with K =10')
[Gm, Pm, wpc, wgc]=margin(GU)   % Returns GM, PH, w_gc, and w_pc
```

```
disp('Uncompensated closed-loop Transfer function')
TU=feedback(GU, 1)
ltiview('step', TU)
z0=0.06; p0=0.0021;            % Controller's zero and pole
Kc= p0/z0;
Gc=tf(Kc*[1  z0], [1  p0])      % Controller transfer function
disp('Compensated open-loop Transfer function')
Gcomp=series(GU, Gc)           % Compensated open-loop transfer function
figure(2)
bode(Gcomp, w); grid on
disp('Compensated GM and PM')
[Gm, Pm, wpc, wgc]=margin(Gcomp)
disp('Compensated closed-loop Transfer function')
Tc=feedback(Gcomp, 1)
ltiview('step', Tc)
```

The result is

```
Uncompensated GM, PM with K =10
        Gm =
                0.2500
        Pm =
                -28.2661
        wpc =
                2.0000
        wgc =
                3.7539
Uncompensated closed-loop Transfer function
               80
        ----------------------
        s^3 + 5 s^2 + 4 s + 80
Phase-lag controller transfer function
        0.035 s + 0.0021
        ----------------
          s + 0.0021
 Compensated open-loop Transfer function:
                 2.8 s + 0.168
        --------------------------------------
        s^4 + 5.002 s^3 + 4.011 s^2 + 0.0084 s
 Compensated GM and PM
        Gm =
                6.6257
        Pm =
                45.1205
        wpc =
                1.9263
```

wgc =

0.5973

Compensated closed-loop Transfer function

Transfer function:

```
              2.8 s + 0.168
---------------------------------------------
s^4 + 5.002 s^3 + 4.011 s^2 + 2.808 s + 0.168
```

Phase-lead Design

Considering a phase-lead controller with a unity dc gain, the transfer function is

$$G_c(s) = \frac{1+\dfrac{s}{\omega_0}}{1+\dfrac{s}{\omega_p}} \quad \text{with } \omega_0 < \omega_p$$

or the frequency response transfer function is

$$G_c(j\omega) = \frac{1+\dfrac{j\omega}{\omega_0}}{1+\dfrac{j\omega}{\omega_p}} \qquad (5.2)$$

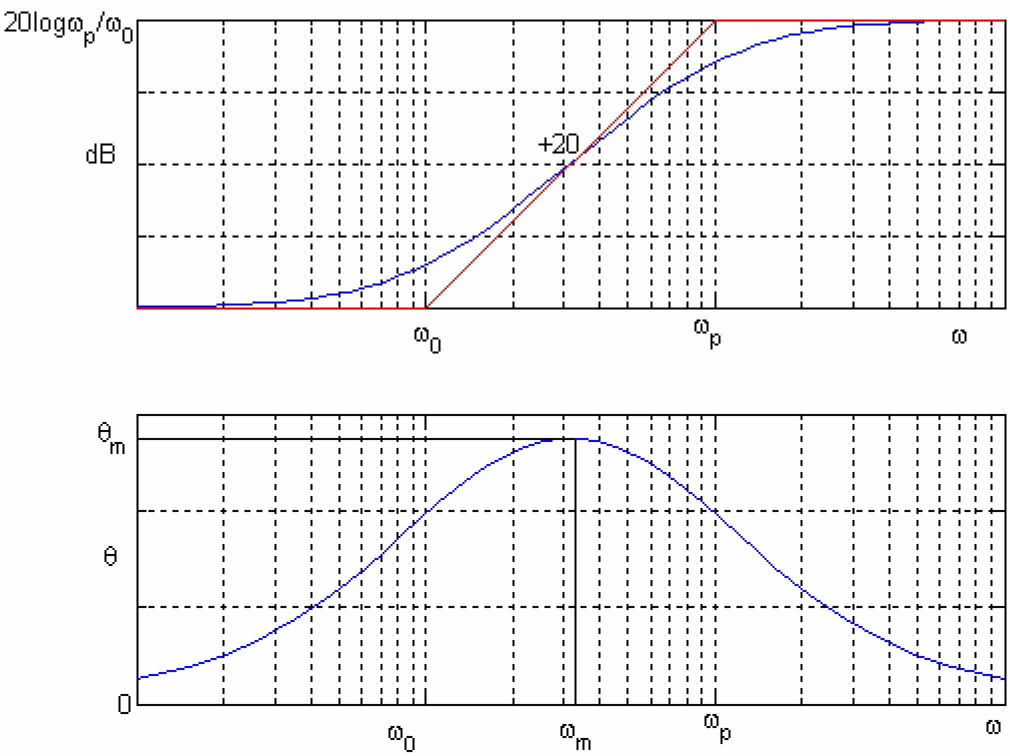

Figure 5.6 Bode diagram for phase-lead controller

The phase-lead controller is a high-pass filter and contributes a positive phase angle, and its inclusion would cause a counter clockwise rotation of the Nyquist diagram away from the –1 point, increasing the phase margin and the gain margin. This type of controller is used to improve the transient characteristics. The maximum phase angle occurs at the geometric mean of ω_0 and ω_p, and is given by

$$\omega_m = \sqrt{\omega_0 \omega_p} \tag{5.3}$$

From (5.2) the controller phase angle is

$$\theta = \tan^{-1}\frac{\omega}{\omega_0} - \tan^{-1}\frac{\omega}{\omega_p} \tag{5.4}$$

It can be shown that the required phase lead ϕ_m is given by

$$\sin\phi_m = \frac{\alpha - 1}{\alpha + 1} \tag{5.5}$$

Where

$$\alpha = \frac{\omega_p}{\omega_0} = \frac{p_0}{Z_0} \tag{5.6}$$

The controller poles and zeros are placed in the vicinity of the 0 dB. To obtain maximum additional phase lead, we desire to place the controller so that the frequency ω_m is located at the compensated gain crossover frequency. For the phase-lead design if the phase margin of the uncompensated system is less than the desired specification. The additional phase margin required is used in (5.5) to evaluate the value of α. The controller total magnitude gain is $20\log_{10}\frac{\omega_p}{\omega_0} = 20\log_{10}\alpha$. Since $\omega_m = \sqrt{\omega_0\omega_p}$ approximately half way between poles and zero, there will be a gain of $10\log_{10}\alpha$ at ω_m. Since both gain and phase margin of the controller affect the phase margin, the graphical phase-lead design a trail-and-error process. The procedure is as follows:

1. For the value of gain that satisfies the steady-state error requirement obtain the Bode plot and determine the uncompensated system's phase margin.

2. Add a small angle to the desired phase angle specified and determine the additional phase angle ϕ_m to be contributed by the controller.
3. Determine α from (5.5).

4. Locate the frequency where the uncompensated magnitude curve is equal to $-10\log\alpha$ dB. This frequency is the new 0 dB crossover frequency.

5. Calculate the pole $\omega_p = \omega_m\sqrt{\alpha}$ and $\omega_0 = \frac{\omega_p}{\alpha}$
6. Draw the uncompensated frequency response and obtain the phase margin and redesign if necessary. You can raise the Amplifier gain to account for the small attenuation introduced by $1/\alpha$

Example 5.2

A feedback control system has an open loop transfer function

$$GH(s) = \frac{K}{s(s+2)}$$

Design a phase-lead such that the system has a phase margin of 45° and a steady-state error of 0.05 due to a ramp input.

$$e_{ss} = \frac{1}{K_v} = 0.05 \quad \Rightarrow \quad K_v = 20$$

The required gain is given by

$$K_v = \lim_{s \to 0} sG_c(s) = \frac{K}{2} = 20 \quad \Rightarrow \quad K = 40$$

The Bode plot for $\frac{K}{s^2 + 2s}$ with $K = 40$ to satisfy the error specification is obtained as shown in Figure 5.7. As it can be seen with this gain $\omega_{gc} < \omega_{pc}$, system is stable with an infinite gain margin and phase margin $\phi_m = 18^{\circ}$

We need to add a phase-lead controller to raise the gain margin to 45° at the new gain crossover frequency

$$\phi_m' = 45 - 18 = 27^{\circ}$$

Allowing about $(5^{\circ} - 10^{\circ})$ 6° to compensate for the lower uncompensated system's phase angle,

$$\phi_m' = 27 + 6 = 33^{\circ}$$

From (5.5)

$$\sin 37 = \frac{\alpha - 1}{\alpha + 1} \quad \Rightarrow \quad \alpha = 3.52$$

From step 4 the compensated gain crossover frequency located at ω_m is located where the uncompensated gain is equal to $-10\log_{10}\alpha = -10\log_{10} 3.52 = -5.46$ dB. From the magnitude plot the frequency is found to be $\omega_m = 8.53$ rad/s.

In step 5 we compute the poles and zeros of the controller

$$\omega_p = \omega_m \sqrt{\alpha} = 8.53\sqrt{3.52} = 16$$

$$\omega_0 = \frac{\omega_p}{\alpha} = \frac{16}{3.52} = 4.55$$

$$K_c = \frac{\omega_p}{\omega_0} = \frac{16}{4.55} = 3.52$$

Thus the controller is

$$G_c(s) = 3.52\frac{s + 4.55}{s + 16}$$

This results in a phase margin of 47°

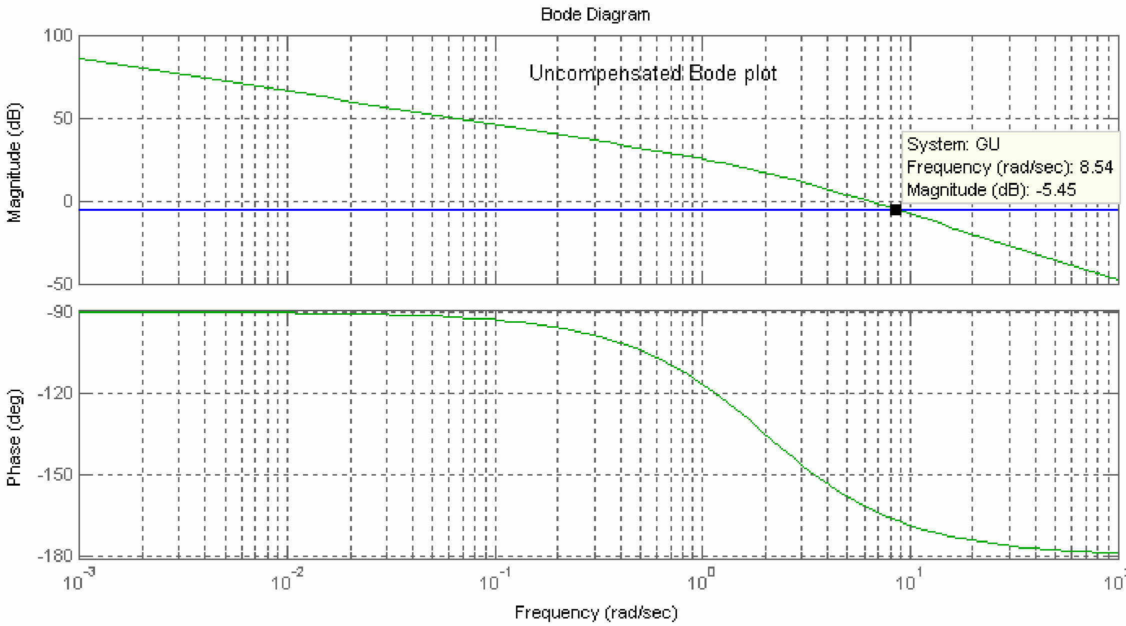

Figure 5.7 Bode diagram for Example 5.2, $K = 40$ to satisfy the steady-state error

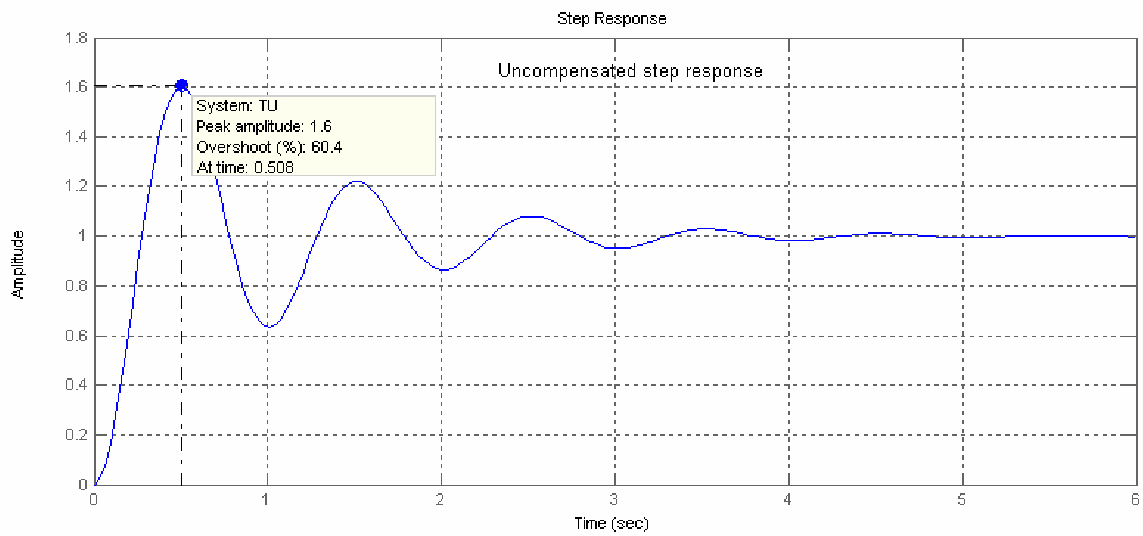

Figure 5.8 Uncompensated step response

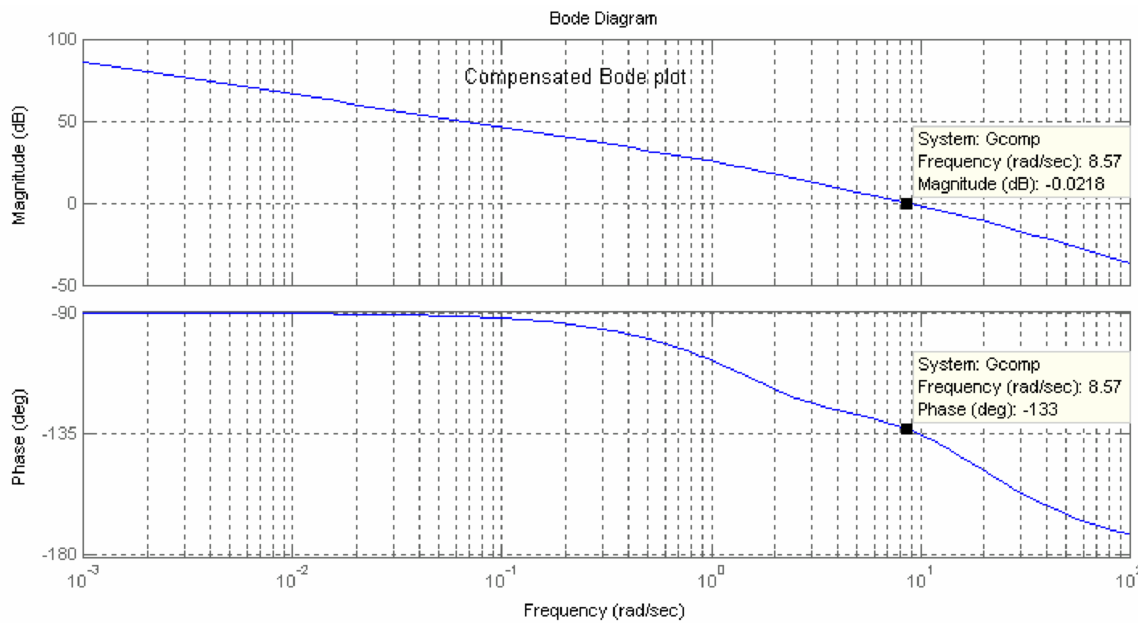

Figure 5.9 The compensated Bode diagram

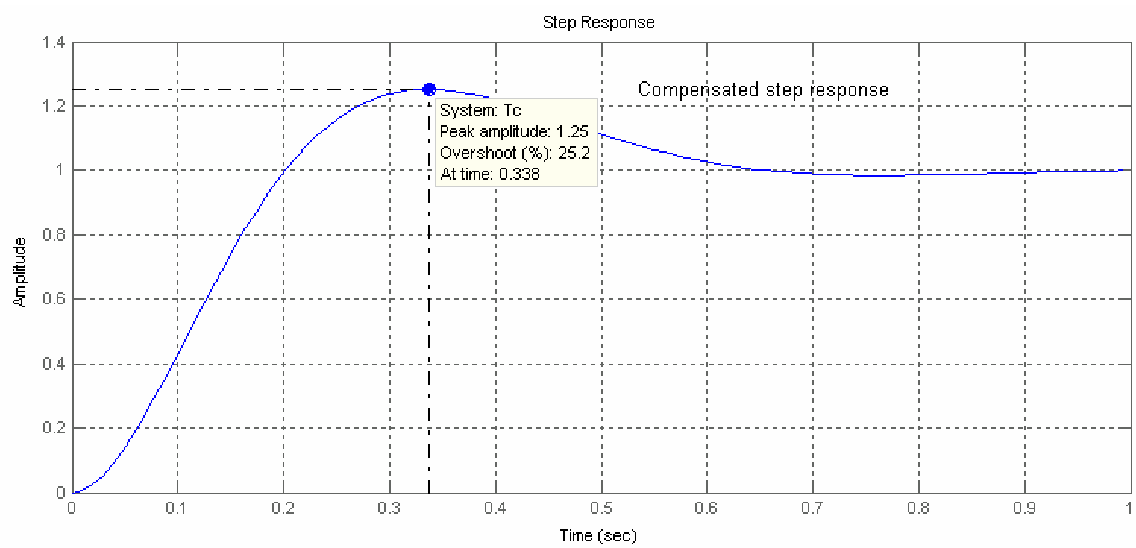

Figure 5.10 Compensated step response

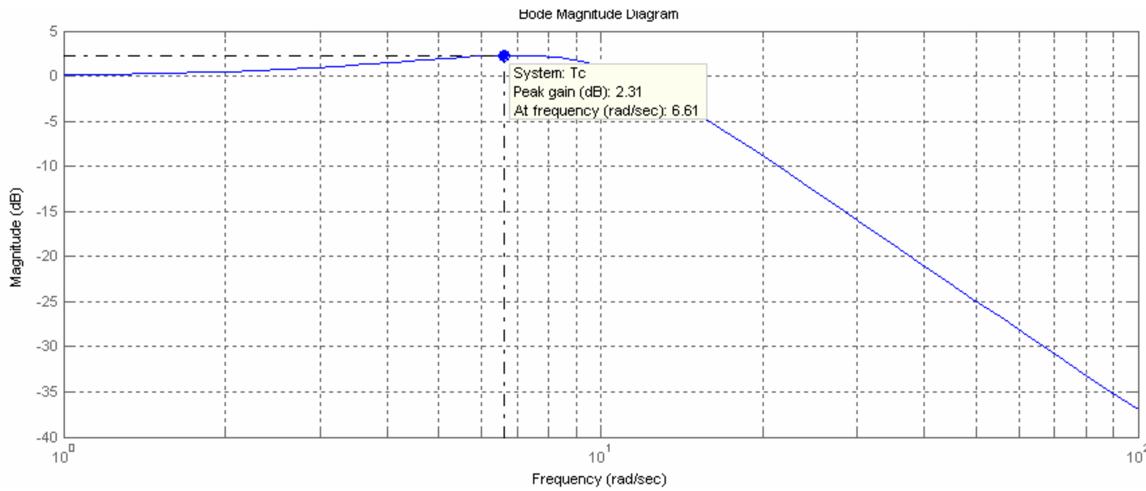

Figure 5.11 Compensated closed-loop frequency response

The following MATLAB commands are used in this Example

```
K=40;
num=1;
den=[1  2 0];
disp('Uncompensated open-loop Transfer function')
GU = tf(num*K, den)           % Open-loop transfer function with K =10
w=logspace(-3,2, 100);        % specified range for w 10^-3 to 10^2
figure(1)
bode(GU, w); grid on          % Returns the Bode magnitude and phase plot
hold on
plot([10^-3  100],[-5.46, -5.46])
disp('Uncompensated closed-loop Transfer function')
TU=feedback(GU, 1)
ltiview('step', TU)
disp('Uncompensated GM, PM with K =10')
[Gm, Pm, wpc, wgc]=margin(GU)   % Returns GM, PH, w_gc, and w_pc
z0=4.55; p0=16;                 % Controller's zero and pole
Kc= p0/z0;

disp('Phase-lead controller transfer function')
Gc=tf(Kc*[1  z0], [1  p0])      % Controller transfer function
disp('Compensated open-loop Transfer function')
Gcomp=series(GU, Gc)            % Compensated open-loop transfer function
figure(2)
bode(Gcomp, w); grid on
disp('Compensated GM and PM')
[Gm, Pm, wpc, wgc]=margin(Gcomp)
disp('Compensated closed-loop Transfer function')
Tc=feedback(Gcomp, 1)
```

```
ltiview('step', Tc)
```

The result is

```
Uncompensated open-loop Transfer function
            40
           ---------
           s^2 + 2 s
Uncompensated closed-loop Transfer function
                40
           --------------
           s^2 + 2 s + 40
Uncompensated GM, PM with K =10
           Gm =
                     Inf
           Pm =
                     17.9642
           wpc =
                      Inf
           wgc =
                     6.1685
Phase-lead controller transfer function
           3.516 s + 16
           ------------
             s + 16
Compensated open-loop Transfer function
             140.7 s + 640
           -------------------
           s^3 + 18 s^2 + 32 s
Compensated GM and PM
           Gm =
                     Inf
           Pm =
                      47.0241
           wpc =
                     Inf
           wgc =
                     8.5516
Compensated closed-loop Transfer function
                140.7 s + 640
           ----------------------------
           s^3 + 18 s^2 + 172.7 s + 640
```