EE-479
Digital Control Systems
Lecture notes

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CHAPTER 1

Modern Control Design

Modern control systems are usually modeled in state-variable form. Modern control design is especially useful in multivariable systems. The simplest design is the state feedback known as *pole-placement design*. The pole-placement design allows all roots of the system characteristic equation to be placed in desired locations. This results in a regulator with constant gain vector K. The state-variable feedback concept requires that all states be accessible.

Pole-Placement Design

Consider the system

Figure 1.1 Open-loop Plant

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) \tag{1.1}$$

The objective is to design the controller that would produce a desired response. Consider the block diagram of the system shown in Figure 1.2 with the following state feedback control

$$u(t) = -Kx(t) \tag{1.2}$$

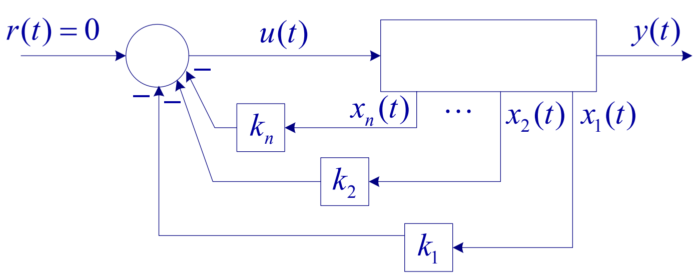


FIGURE 1.2 Control system design via pole placement.

where K is a $1 \times n$ vector of constant feedback gains. The control system input $r(t)$ is assumed to be zero. The purpose of this system is to return all state variables to equilibrium state or zero state when the states have been perturbed.

Substituting (1.2) in (1.1), we have

$$\dot{x}(t) = Ax(t) - BKx(t)$$

$$\begin{aligned}\dot{x}(t) &= [A - BK]x(t) \\ \dot{x}(t) &= A_f x(t)\end{aligned} \tag{1.3}$$

where $A_f = A - BK$

The compensated system characteristic equation is given by $\left|SI - A_f\right| = 0$, or

$$\left|SI - A + BK\right| = 0 \tag{1.4}$$

If the state equation is in phase variable control canonical form, i.e.,

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \cdots & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n1} \\ \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t) \tag{1.5}$$

Then the compensated system characteristic equation becomes

$$\left|SI - A + BK\right| = s^n + (a_{n-1} + k_n)s^{n-1} + \cdots + (a_1 + k_2)s + (a_0 + k_1) = 0 \tag{1.6}$$

The design objective is to find the gain matrix K such that the characteristic equation for the controlled system is identical to the desired characteristic equation obtained by the specified closed-loop poles. For the specified closed-loop pole locations $-\lambda_1, -\lambda_2, \cdots,$ $-\lambda_n$. The desired characteristic equation is

$$\alpha_c(s) = (s + \lambda_1)(s + \lambda_2)\ldots(s + \lambda_n) = s^n + a_{n-1}s^{n-1} + \cdots a_1 s + a_0 = 0 \tag{1.7}$$

Thus, the gain vector K is obtained by equating coefficients of equations (1.6) and (1.7)

$$K_i = \alpha_{i-1} - a_{i-1} \tag{1.8}$$

If the state model is not in the phase-variable canonical form, we can use the transformation technique of Chapter 3 Section 3.5 to transform the given state model to the phase-variable canonical form. The gain factor is obtained for this model and then transformed back to confirm with the original model. This procedure results in the following formula, known as *Ackermann's formula*.

$$K = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} S^{-1} \alpha_c(A) \tag{1.9}$$

where the matrix S known as *controllability matrix* is given by

$$S = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \tag{1.10}$$

and the notation $\alpha_c(A)$ is given by

$$\alpha_c(A) = A^n + \alpha_{n-1}A^{n-1} + \cdots + \alpha_1 A + \alpha_0 I \tag{1.11}$$

The function **[K, *Af*] = placepol(A, B, C, p)** is developed for the pole-placement design. **A, B, C** are system matrices and **p** is a row vector containing the desired closed-loop poles. This function returns the gain vector **K** and the closed-loop system matrix A_f Also, the *MATLAB Control System Toolbox* contains two functions for pole-placement design. Function **K = acker(A, B, p)** is for single input systems, and function **K = place(A, B, p)**, which uses a more reliable algorithm, is for multi-input systems.

The condition that must exist to place the closed-loop poles at the desired location is to be able to transform the given state model into phase-variable canonical form. That is, the controllability matrix S, given in (1.10), must have a nonzero determinant. This characteristic is known as *controllability*.

Example 1.1 (ChD1Ex1.mdl)

A system is described by

$$\ddot{x} + 2\dot{x} - 3x = u(t), \qquad given \qquad x(0) = 1, \; and \;\; \dot{x}(0) = -1$$

Represent the system in state variable control canonical form. Design a state feedback controller to place the closed-loop pole at $-4 \pm j3$.

Let

$$x_1 = x$$
$$x_2 = \dot{x}$$

Then

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = 3x_1 - 2x_3 = u(t)$$

or in matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

The desired characteristic equation is given by

$$\alpha_c(s) = (s + 4 - j3)(s + 4 + j3) = s^2 + 8s + 25 = 0$$

$$\left[SI - A_f\right] = \left[SI - A + BK\right] = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix}$$
$$= \begin{bmatrix} s & -1 \\ -3 + k_1 & s + 2 + k_2 \end{bmatrix}$$

The compensated system characteristic equation is

$$|SI - A + BK| = \begin{vmatrix} s & -1 \\ -3 + k_1 & s + 2 + k_2 \end{vmatrix} = 0$$
$$= s^2 + (2 + k_2)s + (-3 + k_1) = 0$$

Comparing the above characteristic equation with the desired characteristic equation, we get

$$-3 + k_1 = 25 \quad \Rightarrow \quad k_1 = 25 + 3 = 28$$
$$2 + k_2 = 8 \quad \Rightarrow \quad k_2 = 8 - 2 = 6$$

or using the relation given by (1.8), $K_i = \alpha_{i-1} - a_{i-1}$, we get the same results

$$k_1 = \alpha_0 - a_0 = 25 - (-3) = 28$$
$$k_2 = \alpha_1 - a_1 = 8 - 2 = 6$$

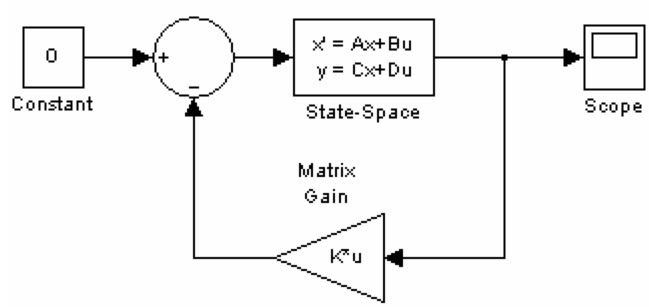
Thus the compensated system is

$$\dot{x}(t) = A_f x(t) = \left[A - BK\right] x(t)$$

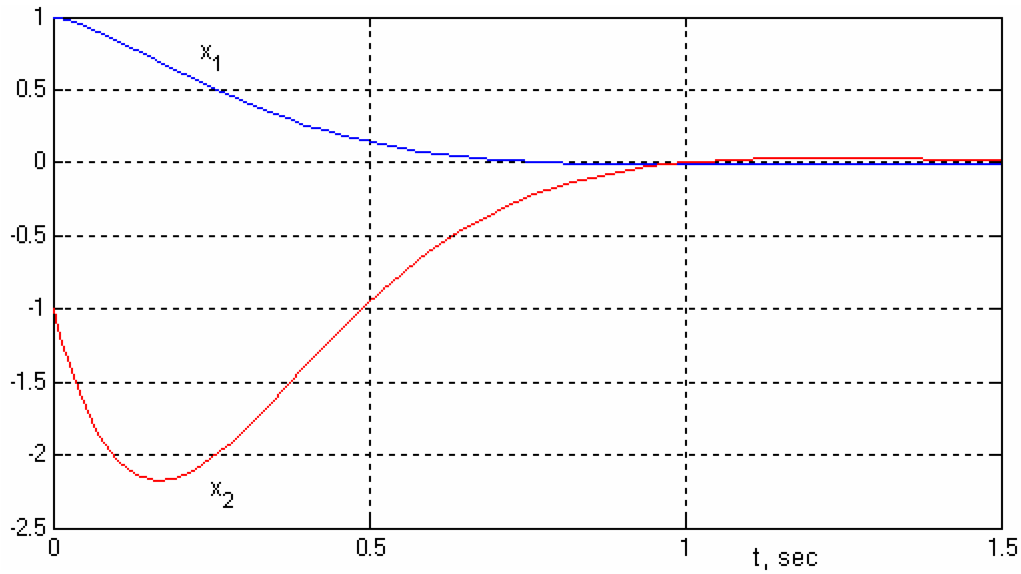
where

$$A_f = \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 28 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -25 & -8 \end{bmatrix}$$

The Simulink block diagram below is used to simulate the response with the given initial condition. The state space block contain the uncompensated matrix A, Column vector B, the 2x2 identity matrix C and the column vector zero of size 2 for D. The initial conditions are set to [1; -1]. The matrix gain is set to [28 6].



The response is as shown.



Example 1.2

A system is described by

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 6 & -1 & -4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}u(t)$$

Design a state feedback controller to place the closed-loop pole at $-3\pm j4$ and -5.

The desired characteristic equation is given by

$$\alpha_c(s)=(s+3-j4)(s+3+j4)(s+5)=(s^2+6s+25)(s+5)=s^3+11s^2+55s+125=0$$

Therefore, $\alpha_0=125,\quad \alpha_1=55,\quad \alpha_2=11$. Also, $a_0=-6,\quad a_1=1,\quad a_2=4$
From (1.8), $K_i=\alpha_{i-1}-a_{i-1}$
Hence

$$k_1=\alpha_0-a_0=125-(-6)=131$$
$$k_2=\alpha_1-a_1=55-1=54$$
$$k_3=\alpha_2-a_2=11-4=7$$

$$A_f=\left[A-BK\right]=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 6 & -1 & -4\end{bmatrix}-\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 131 & 54 & 7\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -125 & -55 & -11\end{bmatrix}$$

Example 1.3 (Example 8.1 Computational Aids in Control Systems using MATLAB) (ChD1Ex3.mdl)

For the plant

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t), \quad given \quad x_1(0) = 1,\ x_2(0) = 1,\ x_3(0) = -1$$

$$y = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Design a state feedback controller to place the closed-loop pole at $-3 \pm j4$ and -8.

The desired characteristic equation is given by

$$\alpha_c(s) = (s+3-j4)(s+3+j4)(s+8) = (s^2+6s+25)(s+8) = s^3+14s^2+73s+200 = 0$$

The given state equation is not in phase variable control canonical form and we use the *Ackermann's formula* given by (1.9).

$$K = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} S^{-1} \alpha_c(A)$$

$$B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad AB = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix},$$

$$A^2 = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ -1 & -2 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 3 & 4 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad A^2B = \begin{bmatrix} -1 & 0 & 0 \\ 3 & 4 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$$

Therefore S as given by (1.10) is

$$S = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 3 \\ 0 & 1 & -1 \end{bmatrix} \quad and \quad S^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0.5 & 1.5 \\ 0 & 0.5 & 0.5 \end{bmatrix}$$

and $\alpha_c(A)$ as given by (1.11) is

$$\alpha_c(A) = A^3 + 14A^2 + 73A + 200I$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ -1 & -2 & 0 \\ 1 & 0 & 0 \end{bmatrix}^3 + 14\begin{bmatrix} -1 & 0 & 0 \\ -1 & -2 & 0 \\ 1 & 0 & 0 \end{bmatrix}^2 73\begin{bmatrix} -1 & 0 & 0 \\ -1 & -2 & 0 \\ 1 & 0 & 0 \end{bmatrix} + 200\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 140 & 0 & 0 \\ -38 & 102 & 0 \\ 60 & 0 & 200 \end{bmatrix}$$

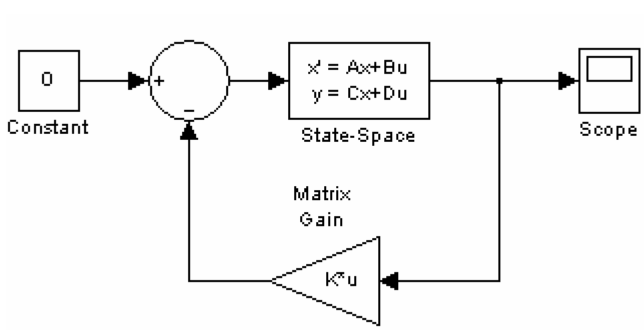
The state feedback matrix as given by (1.9) is

$$K = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0.5 & 1.5 \\ 0 & 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 140 & 0 & 0 \\ -38 & 102 & 0 \\ 60 & 0 & 200 \end{bmatrix} = \begin{bmatrix} 11 & 51 & 100 \end{bmatrix}$$

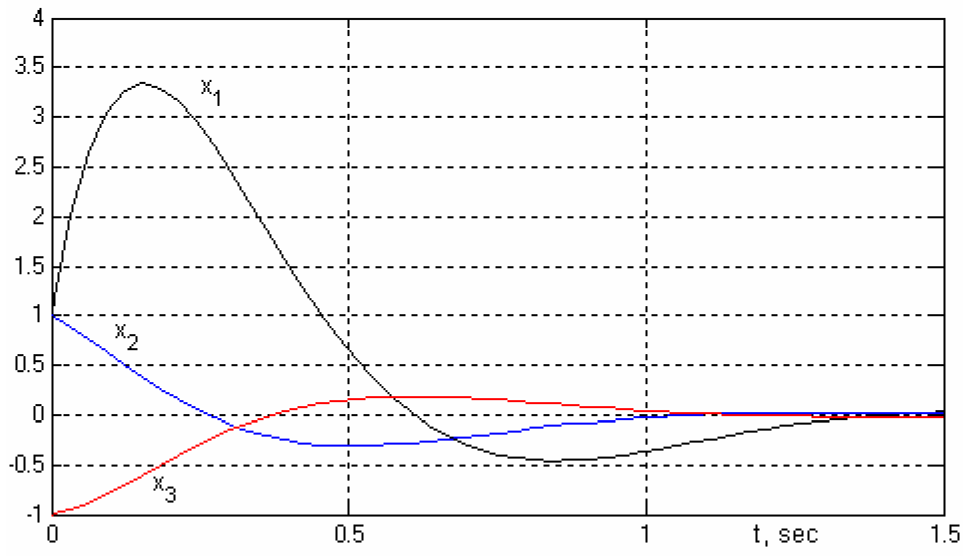
The compensated system matrix $A_f = A = BK$ is

$$A_f = \left[A - BK\right] = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -2 & 0 \\ 1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 11 & 51 & 100 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -12 & -51 & -100 \\ -1 & -2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

The Simulink block diagram below is used to simulate the response with the given initial condition. The state space block contain the uncompensated matrix A, Column vector B, the 3x3 identity matrix C and the column vector zero of size 3 for D. The initial conditions are set to [1; 1; -1]. The matrix gain is set to [11 51 100].



The response is as shown.



For the MATLAB design and the step response see Computational Aids in Control Systems using MATLAB, Example 8.1 page 172.

Controllability

A system is said to be controllable when the plant input u can be used to transfer the plant from any initial state to any arbitrary state in a finite time. The plant described by (1.1) with the system matrix having dimension $n \times n$ is completely state controllable if and only if the controllability matrix **S** in (1.10) has a rank of n. The function **S = cntrable(A, B)** is developed which returns the controllability matrix **S** and determines whether or not the system is state controllable.

Observer Design

In the pole-placement approach to the design of control systems, it was assumed that all state variables are available for feedback. However, in practice it is impractical to install all the sensors which would be necessary to measure all of the states. If the state variables are not available because of system configuration or cost, an observer or estimator may be necessary. The observer is an estimator algorithm based on the mathematical model of the system. The observer creates an estimate $\hat{x}(t)$ of the states from the measurements of the output $y(t)$. The estimated states, rather than the actual states, are then fed to the controller. One scheme is shown in Figure 1.3.

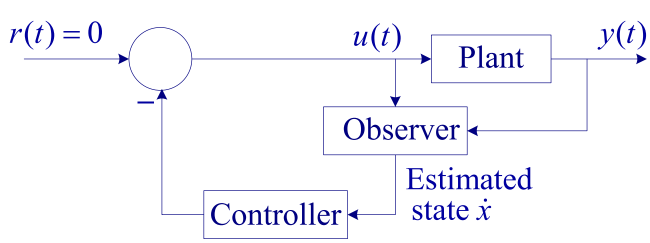


Figure 1.3 State feedback design with an observer

Consider a system represented by the state and output equations

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1.12}$$
$$y(t) = Cx(t) \tag{1.13}$$

Assume that the state $x(t)$ is to be approximated by the state $\hat{x}(t)$ of the dynamic model

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + G\left(y(t) - \hat{y}(t)\right) \tag{1.14}$$
$$\hat{y}(t) = C\hat{x}(t) \tag{1.15}$$

Subtracting (1.14) from (1.12), and (1.15) from (1.13), we have

$$\left(\dot{x}(t) - \dot{\hat{x}}(t)\right) = A\left(x(t) - \hat{x}(t)\right) - G\left(y(t) - \hat{y}(t)\right) \tag{1.16}$$
$$\left(y(t) - \hat{y}(t)\right) = C\left(x(t) - \hat{x}(t)\right) \tag{1.17}$$

where $x(t) - \hat{x}(t)$ is the error between the actual state vector and the estimated vector, and $y(t) - \hat{y}(t)$ is the error between the actual output and the estimated output. Substituting the output equation into the state equation, we obtain the equation for the error between the estimated state vector and the actual state vector.

$$\dot{e}(t) = \left(A - GC\right)e(t) = A_e e(t) \tag{1.18}$$

where

$$e(t) = x(t) - \hat{x}(t) \tag{1.19}$$

If G is chosen such that eigen values of matrix $A - GC$ all have negative real parts, then the steady-state value of the estimated state vector error $e(t)$ for any initial condition will tend to zero. That is, $\hat{x}(t)$ will converge to $x(t)$. The design of the observer is similar to the design of the controller. However, the eigen values of $A - GC$ must be selected to the left of the eigen values of A. This ensures that the observer dynamic is faster than the controller dynamic for providing a rapid updated estimate of the state vector.

The estimator characteristic equation is given by

$$\left|SI - A + GC\right| = 0 \tag{1.20}$$

For a specified speed of response, the desired characteristic equation for the estimator is

$$\alpha_c(s) = s^n + a_{n-1}s^{n-1} + \cdots a_1 s + a_0 = 0 \tag{1.21}$$

Thus, the estimator gain G is obtained by equating coefficients of (1.20) and (1.21). This is identical to the pole-placement technique, and G is found by the application of Ackermann's formula

$$G = \alpha_c(A)\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \tag{1.22}$$

and the notation $\alpha_c(A)$ is given by

$$\alpha_c(A) = A^n + \alpha_{n-1}A^{n-1} + \cdots + \alpha_1 A + \alpha_0 I \tag{1.23}$$

The function **[G,** Ae**] = observer(A, B, C,** p_e**)** is developed for the estimator. p_e is the desired estimator eigen values. This function returns the gain vector **G** and the closed-loop system matrix A_f .

The necessary condition for the design of an observer is that all the states can be observed from the measurements of the output. This characteristic is known as *observability*.

Example 1.4 (Example 8.3 Computational Aids in Control Systems using MATLAB)

For the plant

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 16 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Design a full-state observer such that the observer is critically damped with eigen values at -8, and -8.

The desired characteristic equation is given by

$$\alpha_c(s) = (s+8)(s+8) = s^2 + 16s + 64 = 0$$

$$\alpha_c(A) = A^2 + 16A + 64I = \begin{bmatrix} 0 & 1 \\ 16 & 0 \end{bmatrix}^2 + 16\begin{bmatrix} 0 & 1 \\ 16 & 0 \end{bmatrix} + 64\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 80 & 16 \\ 256 & 80 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad CA = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 16 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$G = \alpha_c(A) \begin{bmatrix} C \\ CA \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 80 & 16 \\ 256 & 80 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 80 & 16 \\ 256 & 80 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 16 \\ 80 \end{bmatrix}$$

Observability

A system is said to be observable if the initial vector x(t) can be found from the measurement of $u(t)$ and $y(t)$. The plant described by (1.12) is completely state observable if the inverse matrix in (1.22) exists. The function **V=obsvable(A, C)** returns the observability matrix **V** and determines whether or not the system is state observable.

Combined Controller-Observer Design

Consider the system represented by the state and output equations (1.12) and (1.13) with the state feedback control based on the observed state $\hat{x}(t)$ given by

$$u(t) = -K\hat{x}(t) \tag{1.24}$$

Substituting for $u(t)$ in (1.12), we have

$$\dot{x}(t) = Ax(t) - BK\hat{x}(t)$$

Substituting for $\hat{x}(t)$ from $e(t) = x(t) - \hat{x}(t)$, the state equation becomes

$$\dot{x}(t) = (A - BK)x(t) + BKe(t) \tag{1.25}$$

Combining the above equation with the error equation given by (1.19), we have

$$\begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - GC \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} \tag{1.26}$$

The function **[K, G, *Ac*] = placeobs(A, B, C, p, *pe*)** is developed for the combined controller-observer design. **A** is the system matrix, **B** is the input column vector, and **C** is the output row vector. **p** is a row vector containing the desired closed-loop poles and *pe* is the desired observer eigen values. The function displays the gain vectors **K** and **G**, open-loop plant transfer function, and the controlled system closed-loop transfer function. Also, the function returns the gain vector **K**, and the combined system matrix in (1.26).

Example 1.5 (Example 8.4 Computational Aids in Control Systems using MATLAB)

For the system of Example 1.4 design a controller-observer system such that the desired closed-loop poles for the system are at $-1 \pm j2$. Choose the eigen values of the observer gain matrix to be $p_{e1} = p_{e2} = -8$.

Controller Design

The desired characteristic equation is given by

$$\alpha_c(s) = (s+1-j2)(s+1+j2) = s^2 + 2s + 5 = 0$$

$\alpha_0 = 5, \quad \alpha_1 = 2$

Since the state equation is in phase variable control canonical form we use the relation given by (1.8), $K_i = \alpha_{i-1} - a_{i-1}$

$$a_0 = -16, \quad a_1 = 0$$

From (1.8), $K_i = \alpha_{i-1} - a_{i-1}$

Hence

$$k_1 = \alpha_0 - a_0 = 5 - (-16) = 21$$
$$k_2 = \alpha_1 - a_1 = 2 - 0 = 2$$

or

$$K = \begin{bmatrix} 21 & 2 \end{bmatrix}$$

Observer Design

The observer was designed in Example 4

$$G = \begin{bmatrix} 16 \\ 80 \end{bmatrix}$$

The combined controller observer system matrix is given by 26

$$\begin{bmatrix} A-BK & BK \\ 0 & A-GC \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 & 1 \\ 16 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 21 & 2 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 21 & 2 \end{bmatrix} \\ 0 & \begin{bmatrix} 0 & 1 \\ 16 & 0 \end{bmatrix} - \begin{bmatrix} 16 \\ 80 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -5 & -2 & 21 & 2 \\ 0 & 0 & -16 & 1 \\ 0 & 0 & -64 & 0 \end{bmatrix}$$

For the MATLAB design and the step response see Computational Aids in Control Systems using MATLAB, Example 8.1 page 177.

Optimal Regulator Design

Refer to computational Aids in Control Systems using MATLAB Section 8.6 page 180.

Transforming the combined controller-observer state equation to transfer function model

The dynamic equation of the observer model in Figure 1.3 was given by equations (1.14) and (1.15) as

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + G\left(y(t) - \hat{y}(t)\right) \tag{1.27}$$

$$\hat{y}(t) = C\hat{x}(t) \tag{1.28}$$

and the control law is given by

$$u(t) = -K\hat{x}(t) \tag{1.29}$$

Substituting for $\hat{y}(t)$ from (1.28) and $u(t)$ from (1.29) into (1.27) results in

$$\dot{\hat{x}}(t) = [A - BK - GC]\hat{x}(t) + Gy(t) \tag{1.30}$$

or

$$\dot{\hat{x}}(t) = A_{ce}\hat{x}(t) + Gy(t) \tag{1.31}$$

where

$$A_{ce} = [A - BK - GC] \tag{1.32}$$

Taking Laplace transform of (1.31) and (1.29), we have

$$sX(s) = A_{ce}X(s) + GY(s) \tag{1.33}$$

$$U(s) = -KX(s) \tag{1.34}$$

Solving for $X(s)$ from (1.33)

$$X(s) = [sI - A_{ce}]^{-1}GY(s) \tag{1.35}$$

Substituting for $X(s)$ in (1.34) results in

$$U(s) = -K[sI - A_{ce}]^{-1}GY(s)$$

or

$$\frac{U(s)}{Y(s)} = -K[sI - A_{ce}]^{-1}G = -G_{ce}(s)$$

Therefore

$$G_{ce} = \frac{Y(s)}{U(s)} = K[sI - A_{ce}]^{-1}G$$

Substituting for A_{ce}, the combined controller observer transfer function is

$$G_{ce} = K[sI - A + BK + GC]^{-1}G \tag{1.36}$$

We can now draw the block diagram representation of the above equation, where the minus sign in $\frac{Y(s)}{U(s)} = -G_{ce}(s)$ is incorporated in the summing point.

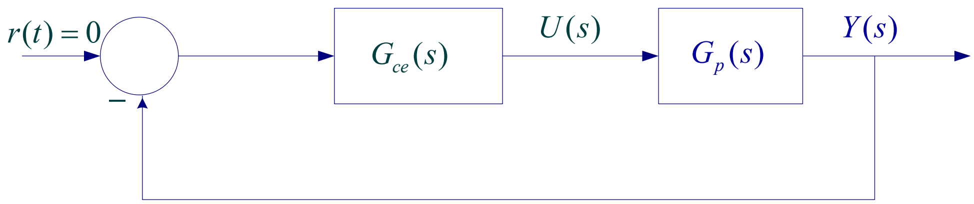


Figure 1.4 Cascade representation of the controller-estimator

Given an input $r(t)$, the closed loop transfer function becomes

$$\frac{R(s)}{Y(s)} = \frac{G_{ce}(s)G_p(s)}{1+G_{ce}(s)G_p(s)} \tag{1.37}$$

Therefore the closed-loop characteristic equation is

$$1+G_{ce}(s)G_p(s) = 0 \tag{1.38}$$