CHAPTER 2

DIGITAL CONTROL SYSTEMS

Introduction

Digital control of a continuous-time system has become very popular as the price and reliability of digital computers has greatly improved. Analog controllers are replaced by digital computer that performs calculations, which emulate the physical controllers. Very complicated control structures can be implemented easily using a digital controller, whereas an analog controller would require very complex hardware. Digital control offers important advantages in flexibility of modifying controller characteristics by changing the program if the design requirement changes or plant dynamic changes with the operating conditions. Furthermore, analog emulation and real-time control provides advanced features such as adaptive self-tuning, multivariable control, expert systems, and the ability to communicate over local area network.

A digital computer may be used to serve as a controller as well as a supervisory. Analog controllers are replaced by digital computer that performs calculations that emulate the physical controllers

Advantages of Digital Control Systems

The use of digital computers in the loop has the following advantages:

- Reduced cost
- Flexibility in response to design changes with many controlled variables
- Noise Immunity
- Ability to communicate over local area network.

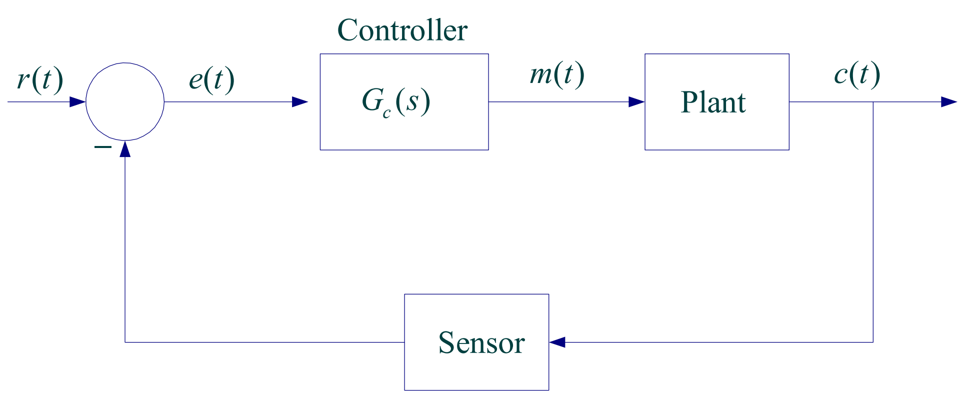


Figure 2.1(a) Continuous system block diagram

Figure 2.1(a) shows the block diagram of the typical continuous system. The computation of the error signal $e(t)$ and the dynamic compensation $G_c(s)$ can all be accomplished in a digital computer as shown in Figure 2.1(b). Typically, the computer replaces the cascade controller as shown in Figure 2.1(a). The measurements data are converted from analog form to digital by means of the analog-to-digital converter A/D). The digital computer receives and performs the required compensation on the signal in digital form (numerical). The computer output is then converted to analog form by the digital-to-analog converter (D/A). A block diagram representation of a digital control system is shown in Figure 2.1(b)

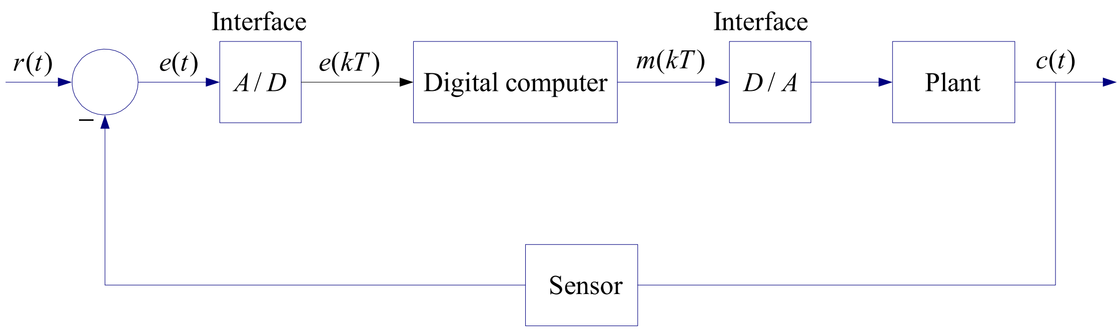


Figure 2.1(b) Digital control system block diagram.

Digital-to-Analog Conversion

D/A converter is a device that converts the sampled signal $m(kT)$ to a continuous signal $m(t)$. The weighted voltages are summed together to produce the analog output.

Analog-to-digital Conversion

In the analog-to-digital converter, the analog signal is first converted to a sampled signal and then converted to a sequence of binary numbers, the digital signal. The sampling rate must be at least twice the bandwidth of the signal, or else there will be distortion. The minimum sampling frequency is known as the *Nyquist sampling rate*.

The output of the A/D converter is described by a number sequence $e(0),\ e(T),\ e(2T),\ \cdots$, designated by $e(kT)$. Often for simplicity, k is omitted, and the notation becomes $e(k)$. Assuming computer has negligible computational time, an input $e(0)$ at time $t=0$, results in an output, which may be expressed by

$$m(0) = b_0 e(0) \tag{2.1}$$

If b_0 is constant, the above equation is linear, time-invariant relation. Depending on the compensation function $m(T),\ m(2T),\ \cdots,\ m(kT)$, or $m(1),\ m(2),\ \cdots,\ m(k)$, may be expressed as

$$m(k) = b_0 e(k) + b_1 e(k-1) + \cdots + b_n e(k-n) - a_1 m(k-1) - \cdots - a_n m(k-n) \tag{2.2}$$

This is called a *difference equation*. Just as differential equations are used to represent systems with analog signals, difference equations are used for system with discrete data. Difference equations are also used to approximate differential equations.

The z-transform

The z-transform is an operational method for solving the difference equation of a discrete linear system. The z-transform plays the same role for discrete-time signals as does the Laplace transform for continuous-time signals. The simplest model for the sampling of the A/D converter is a switch, which repeatedly closes for every short duration τ after every T seconds. The output of such a switch would consist of series of pulses separated by T seconds. If τ is very small the output can be represented by a series of time-shifted impulses. As we see in the next chapter an ideal model of a sampled signal can be presented by

$$x^*(t) = \sum_{k=-\infty}^{k=\infty} x(t)\delta(t-kT) \tag{2.1}$$

Since $\delta(t-kT)$ is zero except at the sampling instant $t = kT$, $x(t)$ can be replaced by $x(kT)$. Also assuming $x(t) = 0$ for $t < 0$, the above equation is written as

$$x^*(t) = \sum_{k=0}^{k=\infty} x(kT)\delta(t-kT) \tag{2.2}$$

Taking the Laplace transform yields

$$X^*(s) = \int_0^\infty \sum_{k=0}^{k=\infty} x(kT)\delta(t-kT)e^{-st}dt$$

Interchanging integral and summation results in

$$X^*(s) = \sum_{k=0}^{k=\infty} x(kT)\int_0^\infty \delta(t-kT)e^{-st}dt$$

From the sifting property of the δ function, the above relation reduces to

$$X^*(s) = \sum_{k=0}^{k=\infty} x(kT)e^{-skT}$$

Now, defining the complex variable z as

$$z = e^{sT} \tag{2.3}$$

we get

$$X(z) = \mathcal{Z}\{x(kT)\} = \sum_{k=0}^{k=\infty} x(kT)z^{-k} \tag{2.4}$$

$X(z)$ is known as the z transform of the sequence of samples, $x(kT)$.
Often T is omitted and the notation becomes

$$X(z) = \mathcal{Z}\{x(k)\} = \sum_{k=0}^{k=\infty} x(k)z^{-k} \qquad k = 0,\ 1,\ 2,\ \cdots$$

The above equation defines the z-transform, and we write

$$x(k) \rightleftarrows X(z) \qquad (2.5)$$

In (2.4) the coefficient $x(kT)$, denotes the sample value, and z^{-k} denotes that the sample occurs k sample periods after the $t=0$ reference. Comparing with (2.3), we see that the parameter z^{-k} is simply shorthand notation for the Laplace time shift operator e^{-skT}. As an example, $20.5z^{-12}$ denotes a sample, having value 20.5, which occurs 12 sample periods after $t=0$ reference.

The Laplace variable s is given by, $s=\sigma+j\omega$ and (2.3) can be written as

$$z=e^{\sigma T}e^{j\omega T}$$

so that the magnitude of z is given by

$$|z|=e^{\sigma T}$$

Thus, the right-half s-plane, $\sigma>0$, corresponds to $z>1$, while the left-half s-plane, $\sigma<0$, corresponds to $z<1$. We see that the left-half s-plane maps into the interior of the unit circle in the z-plane and that the right-half s-plane maps outside the unit circle in the z-plane. The mapping of the Laplace variable s into the z-plane through $z=e^{sT}$ is illustrated in Figure 2.1.

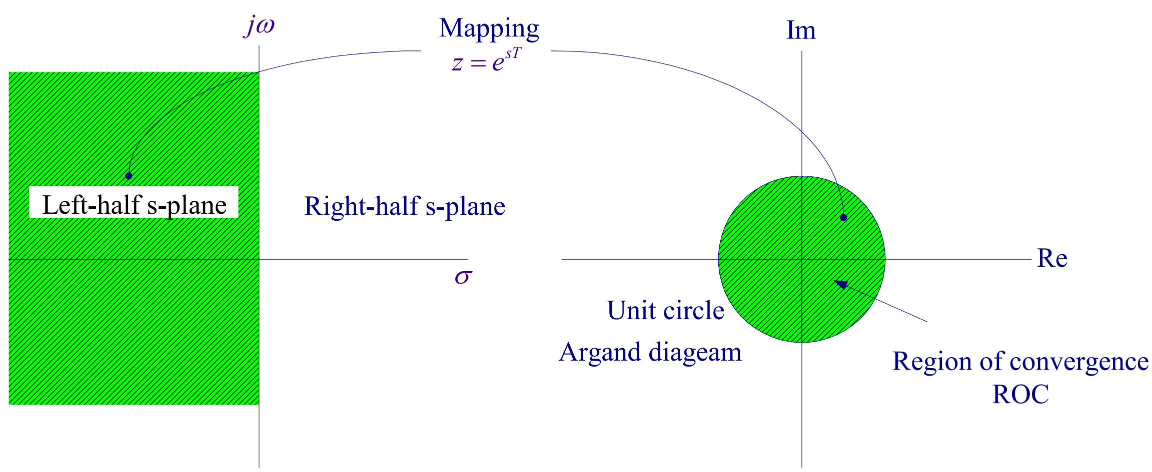


Figure 2.1 Mapping variable s into z-plane

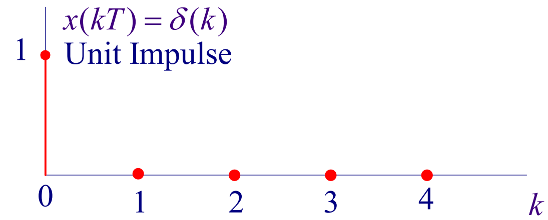
Recall that the s-domain transfer function with poles in the left-half-s-plane resulted a time domain response that decayed to zero as $t\to\infty$. In a similar manner function of z having poles with magnitude less than one decay to zero in the time domain as $k\to\infty$. Poles on the $j\omega$ axis in the s-plane corresponds to poles on the unit circle in the z-plane, and imply time-domain functions that oscillates, i.e marginally stable. An unstable system would have poles in the right-half s-plane, this corresponds to z-domain function with poles outside the unit circle. A function having pole at origin, i.e., $\frac{1}{s}$ is a unit step in the time-domain, and as we see a function having a pole in the z-plane at $z=1$ corresponds to a sampled unit step in the time-domain. This is not surprising since $s=0$ corresponds to $z=1$.

The following Examples illustrate the derivation of the z-transform

Example 2.1

Find the z-transform of the discrete delta function

$$x(kT) = \delta(kT) = \begin{cases} 1 & k = 0 \\ 0 & \text{otherwise} \end{cases}$$



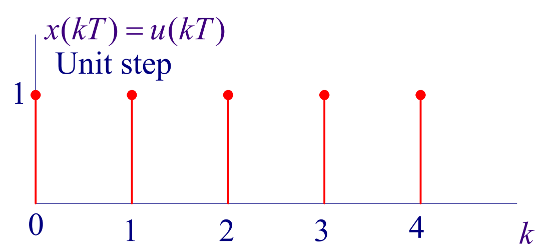
$$X(z) = Z\{\delta(kT)\} = \sum_{k=0}^{k=\infty} \delta(kT) z^{-k} = z^0 = 1 \qquad \Rightarrow \quad \mathcal{Z}\{\delta(kT)\} = 1$$

Recall that the Laplace transform of $\delta(t)$ is also 1. Thus the discrete delta function, like its continuous counterpart, represents the instantaneous injection of energy into a system.

Example 2.2

Find the z-transform of the discrete unit step function

$$x(kT) = u(kT) \qquad k = 0,\ 1,\ 2,\ \cdots$$



$$X(z) = \sum_{k=0}^{k=\infty} u(kT) z^{-k} = \sum_{k=0}^{k=\infty} z^{-k} = 1 + z^{-1} + z^{-2} + z^{-3} + \cdots$$

The following infinite geometric series in closed form is

$$1 + z^{-1} + z^{-2} + z^{-3} + \cdots = \frac{1}{1 - z^{-1}} = \frac{z}{z - 1}$$

That is,

$$\sum_{k=0}^{k=\infty} z^{-k} = \frac{1}{1 - z^{-1}} = \frac{z}{z - 1} \qquad (2.5)$$

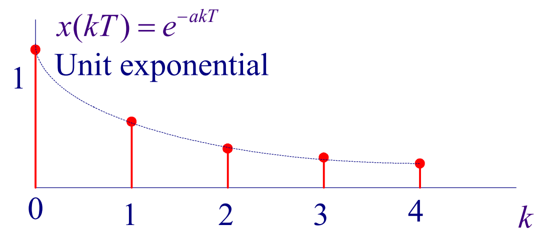
Therefore

$$X(z) = \mathcal{Z}\{u(t)\} = \frac{z}{z-1}$$

Example 2.3

Find the z-transform of the discrete exponential function

$x(kT) = e^{-akT} \qquad k = 0,\ 1,\ 2,\ \cdots$



$$X(z) = \mathcal{Z}\left\{e^{-akT}\right\} = \sum_{k=0}^{k=\infty} e^{-akT} z^{-k} = \sum_{k=0}^{k=\infty} \left(ze^{aT}\right)^{-k}$$

Form (2.8), we conclude that

$$X(z) = \mathcal{Z}\{e^{-aKT}\} = \frac{1}{1-(ze^{aT})^{-1}} = \frac{z}{z-e^{-aT}}$$

e^{-aT} is constant, letting $c = e^{-aT}$, gives the z-transform pair

$$\mathcal{Z}(c^k) = \frac{1}{1-cz^{-1}} = \frac{z}{z-c}$$

which shows that $X(z)$ has a zero at $z = 0$ and a pole at $z = c$. Thus, only for $|c| < 1$, the response is bounded.

Example 2.4

Find the z-transform of the discrete sinusoidal function $x(t) = \cos \omega t$ for $t > 0$

$$\cos \omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2} \quad \text{or} \quad \cos \omega kT = \frac{e^{j\omega kT} + e^{-j\omega kT}}{2}$$

$$X(z) = \frac{1}{2}\left[\frac{z}{z-e^{j\omega T}} + \frac{z}{z-e^{-j\omega T}}\right] = \frac{1}{2}\frac{z(z-e^{-j\omega T} + z - e^{j\omega T})}{z^2 - ze^{-j\omega T} - ze^{j\omega T} + 1}$$

or

$$X(z) = \frac{z(z-\cos \omega T)}{z^2 - 2z\cos \omega T + 1}$$

The z-transform for typical discrete function along with the corresponding Laplace transform for continuous function is given in table 2.1.

Table 2.1

$x(t)$	$X(s)$	$x(kT)$	$X(z)$
$\delta(t)$	1	$\delta(kT)$	1
$u(t)$	$\frac{1}{s}$	$u(kT)$	$\frac{z}{z-1}$
t	$\frac{1}{s^2}$	kT	$\frac{Tz}{(z-1)^2}$
t^2	$\frac{2}{s^3}$	$(kT)^2$	$\frac{T^2 z(z+1)}{(z-1)^3}$
e^{-at}	$\frac{1}{s+a}$	$e^{-kaT} = (e^{-aT})^k = c^k$	$\frac{z}{z-e^{-aT}} = \frac{z}{z-c}$
te^{-at}	$\frac{1}{(s+a)^2}$	kTe^{-kaT}	$\frac{Tze^{-aT}}{(z-e^{-aT})^2}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$	$\sin \omega kT$	$\frac{z \sin \omega T}{z^2 - 2z\cos \omega T + 1}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$	$\cos \omega kT$	$\frac{z(z-\cos \omega T)}{z^2 - 2zco\omega T + 1}$
$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$	$e^{-aT} \sin \omega kT$	$\frac{ze^{-aT} \sin \omega T}{z^2 - 2ze^{-aT} \cos \omega T + e^{-2aT}}$
$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$	$e^{-aT} \cos \omega kT$	$\frac{z^2 - ze^{-aT} \cos \omega T}{z^2 - 2ze^{-aT} \cos \omega T + e^{-2aT}}$

Initial and Final value theorems

The initial and final value theorems are useful for providing partial check on the time-domain response without having to compute the inverse z-transform. The initial value theorem states that

$$x(0) = \lim_{z \to \infty} X(z) \tag{2.6}$$

This is easily verified. By definition

$$X(z) = \sum_{k=0}^{\infty} x(kT) z^{-k} = x(0) + \sum_{k=1}^{\infty} x(kT) z^{-k}$$

as $z \to \infty$, the summation on the right vanishes, and (2.6) is verified.

The final value theorem states that

$$x(\infty) = \lim_{z \to 1}(1 - z^{-1})X(z) \tag{2.7}$$

or

$$x(\infty) = \lim_{z \to 1}(z - 1)X(z) \tag{2.9}$$

The proof of this theorem is more involved and will not be discussed here. However this can easily be justified. If $X(z)$ has any pole outside the unit circle, $X(z)$ correspond to an unstable function and $x(\infty) = \infty$. If $X(z)$ has all its poles inside the unit circle, $X(z)$ correspond to a stable function and $x(\infty) = 0$. If $X(z)$ has a pole on the unit circle but not a t $z = 1$, the resulting time-domain response is oscillatory, and $x(\infty)$ is not defined. A nonzero steady-state value results from a simple pole at $z = 1$. A list of some of the z-transform theorem is given in Table 2.2

Real translation – Time delay

If the time sequence $x(kT)$ is delayed by n sample periods, the z-transform $X(z)$ is multiplied by z^{-n}, i.e.,

$$\begin{aligned}
&\mathcal{Z}\{x(k)T\} = X(z) && n = 0 \quad \text{This is simlar to } \mathcal{L}[x(t)] = X(s) \\
&\mathcal{Z}\{x(k-1)T\} = z^{-1}X(z) && n = 1 \quad \text{This is simlar to } \mathcal{L}[\textstyle\int x(t)dx] = \frac{1}{s}X(s) \\
&\mathcal{Z}\{x(k-2)T\} = z^{-2}X(z) && n = 2 \quad \text{etc} \\
&\vdots \\
&\mathcal{Z}\{x(k-n)T\} = z^{-n}X(z)
\end{aligned} \tag{2.10}$$

This can easily be verified. Since the z-transform of the sequence $x(k-n)T$ is

$$\mathcal{Z}\{x(k-n)T\} = \sum_{n=0}^{\infty} x(k-n)Tz^{-k}$$

Letting $m = k - n$, we have

$$\mathcal{Z}\{x(k-n)T\} = \sum_{m=-n}^{\infty} x(m)Tz^{-m-n}$$

Since $x(mT)$ is assumed zero for $m < 0$, we change the lower limit in the above summation to $m = 0$. Rewriting this expression, we obtain

$$\mathcal{Z}\{x(k-n)T\} = z^{-n}\sum_{m=0}^{\infty} x(mT)z^{-m} = z^{-n}X(z)$$

Real translation – Time advance

In a similar way the time advance shift is obtained and is as follow

$$\mathcal{Z}\{x(k+n)T\} = z^{n}[X(z)-\sum_{k=0}^{n-1} x(kT)z^{-k}] \tag{2.11}$$

This relation is very important in transforming the difference equation into z-domain and solving for the z-domain variable. The above operator for $n=1$ and $n=2$, become

$$\mathcal{Z}\{x(k+1)\} = zX(z) - zx(0) \tag{2.12}$$

$$\mathcal{Z}\{x(k+2)\} = z^2X(z) - z^2x(0) - zx(1) \tag{2.13}$$

These operational transforms along with the linearity property, multiplication by exponential and multiplication by time are summarized in Table 2.2

Table 2.2 Operational transform

Theorem	s-domain	z-domain
Linearity theorem	$\mathcal{L}\{x(t)\} = X(s)$ $\mathcal{L}\{ax(t)\} = aX(s)$	$\mathcal{Z}\{(kT)\} = X(z)$ $\mathcal{Z}\{ax(kT)\} = aX(z)$
Linearity theorem	$\mathcal{L}\{x_1(t)+x_2(t)\} = X_1(s)+X_2(s)$	$\mathcal{Z}\{x_1(kT)+x_2(kT)\} = X_1(z)+X_2(z)$
Multiplication by exponential	$\mathcal{L}\{e^{-at}x(t)\} = X(s+a)$	$\mathcal{Z}\{e^{-akT}x(T)\} = X(ze^{aT})$
Multiplication by t	$\mathcal{L}\{tx(t)\} = -\frac{dX(s)}{ds}$	$\mathcal{Z}\{kTx(kT)\} = -Tz\frac{dX(z)}{dz}$
Initial value theorem	$x(0) = \lim_{s\to\infty} sX(s)$	$x(0) = \lim_{z\to\infty} X(z)$
Final value theorem	$x(\infty) = \lim_{s\to 0} sX(s)$	$x(\infty) = (1-z^{-1})\lim_{z\to 1} X(z)$ or $x(\infty) = (z-1)\lim_{z\to 1} X(z)$
Real-translation, time delay		$\mathcal{Z}\{x(k-n)T\} = z^{-n}X(z)$
Real-translation, time advance		$\mathcal{Z}\{x(k+n)T\} = z^{n}[X(z)-\sum_{k=0}^{n-1} x(kT)z^{-k}]$
Real-translation, time advance (order n=1)		$\mathcal{Z}\{x(k+1)\} = zX(z) - zx(0)$
Real-translation, time advance (order n=2)		$\mathcal{Z}\{x(k+2)\} = z^2X(z) - z^2x(0) - zx(1)$

Example 2.5

Find the z-transform of the sampled unit ramp function $tu(t)$ for $t > 0$

From the multiplication by theorem $\mathcal{Z}\{tx(t)\} = -Tz\dfrac{dX(z)}{dz}$

$$\mathcal{Z}\{tu(t)\} = -Tz\frac{dX(z)}{dz}$$

$$= -Tz\frac{d}{dz}\frac{z}{z-1} = -Tz\frac{-1}{(z-1)^2}$$

Therefore,

$$X(z) = \mathcal{Z}\{tu(t)\} = \frac{Tz}{(z-1)^2}$$

Example 2.6

Find the z-transform of the sampled function $x(t) = e^{-3t}\sin 4t$

We know that $\mathcal{Z}[\sin 4kT] = \dfrac{z\sin 4T}{z^2 - 2z\cos 4T + 1}$

From multiplication by exponential theorem $\mathcal{Z}\{e^{-at}x(t)\} = X(ze^{aT})$, replacing z by ze^{3t}, we have

$$\mathcal{Z}[e^{-3T}\sin 4kT] = \frac{ze^{3T}\sin 4T}{z^2e^{6T} - 2ze^{3T}\cos 4T + 1} = \frac{ze^{-3T}\sin 4T}{z^2 - 2ze^{-3T}\cos 4T + e^{-6T}}$$

MATLAB symbolic Math Toolbox provide **ztrans** function for performing the z-transform.

Example 2.7

Use MATLAB to find the z-transform of the sampled function $x(t) = e^{-at}\sin bt$ for $t > 0$.
We use the following commands

```
syms  k z a b T
xk=exp(-a*T*k)*sin(b*T*k);
Xz=ztrans(xk);
```

The result is

```
Xz =
        z/exp(-a*T)*sin(b*T)/(z^2/exp(-a*T)^2-2*z/exp(-a*T)*cos(b*T)+1)
```

Upon multiplying the numerator and denominator of the above expression by e^{-2aT}, we obtain the same result as given in Table 2.1.

The Inverse z-Transform

There are two methods for finding the inverse z-transform (1) partial fraction expansion and (2) the power series method. Since the z-transform came from the discrete function, its inverse transform even though it may be expressed in closed-form the result is valid only at the sampling instants.

Inverse z-transform via partial fraction expansion

Recall that the Laplace transform consists of a partial fraction that yields a sum of terms leading to exponentials, that is $A/(s+a)$. Taking this lead and noting that the discrete time functions are related to their z-transform as follows:

$$\mathcal{Z}[e^{-akT}] = \frac{z}{z-e^{-aT}} \quad \Rightarrow \quad \mathcal{Z}^{-1}\left[\frac{z}{z-c}\right] = c^k$$

We thus predict that a partial fraction expansion should be of the following form:

$$X(z) = \frac{Az}{z-z_1} + \frac{Bz}{z-z_2} + \cdots$$

Since the partial fraction expansion of $X(s)$ did not contain terms with s the numerator of the partial fraction expansion, we first form $X(z)/z$ to eliminate the z term in the numerator, perform a partial fraction expansion of $X(z)/z$, and finally multiply the result by z to replace the z's in the numerator.

Example 2.8

Find the inverse z-transform of

$$X(z) = \frac{3z}{(z-1)(z-0.4)}$$

First we find $\dfrac{X(z)}{z}$ and then perform the partial fraction expansion

$$\frac{X(z)}{z} = \frac{3}{(z-1)(z-0.4)} = \frac{5}{(z-1)} + \frac{-5}{(z-0.4)}$$

Multiplying by z, we have

$$X(z) = \frac{5z}{(z-1)} + \frac{-5z}{(z-0.4)}$$

Using the z-transform pair from Table 2.1, the inverse z-transform is

$$x(kT) = 5 - 5(0.4)^k$$

From the above expression, we see that

$$x(0) = 5 - 5(0.4)^0 = 0$$
$$x(1) = 5 - 5(0.4)^1 = 3$$
$$x(2) = 5 - 5(0.4)^2 = 4.2$$
$$x(3) = 5 - 5(0.4)^3 = 4.68$$
$$\vdots$$
$$x(\infty) = 5$$

We can check $x(0)$, and $x(\infty)$ from the initial and final value theorem

$$x(0) = \lim_{z\to\infty} X(z) = \lim_{z\to\infty} \frac{3z}{z^2} = 0 \text{ and } x(\infty) = \lim_{z=1}(z-1)X(z) = \frac{3}{1-0.4} = 5$$

Example 2.9

Find the inverse z-transform of $X(z) = \dfrac{(1-e^{-aT})z}{(z-1)(z-e^{-aT})}$

First we find $\dfrac{X(z)}{z}$ and then perform the partial fraction expansion

$$\frac{X(z)}{z} = \frac{(1-e^{-aT})}{(z-1)(z-e^{-aT})} = \frac{1}{(z-1)} + \frac{-1}{(z-e^{-aT})}$$

Multiplying by z, we have

$$X(z) = \frac{z}{(z-1)} + \frac{-z}{(z-e^{-aT})}$$

Using Table 2.1, the inverse z-transform is

$$x(kT) = 1 - e^{-akT}$$

Example 2.10

Find the inverse z-transform of $X(z) = \dfrac{1}{(z-1)(z-0.5)}$

First we find $\dfrac{X(z)}{z}$ and then perform the partial fraction expansion

$$\frac{X(z)}{z} = \frac{1}{z(z-1)(z-0.5)} = \frac{2}{z} + \frac{2}{(z-1)} + \frac{-4}{(z-0.5)}$$

Multiplying by z, we have

$$X(z) = 2 + \frac{2z}{(z-1)} + \frac{-4z}{(z-0.5)}$$

Using Table 2.1, the inverse z-transform is

$$x(kT) = 2\delta(kT) + 2 - 4(0.5)^k$$

MATLAB Symbolic Toolbox provides the function **iztrans** for finding the inverse z-transform.

Example 2.11

Using MATLAB Symbolic Toolbox find the inverse z-transform of the function of Example 2.10.

We use the following commands

```
syms  k z
Xz=1/(z^2-1.5*z+0.5);
xk=iztrans(Xz,k)
```

The result is

```
xk =
      2*charfcn[0](k)+2-4*(1/2)^k
```

which is the same result as Example 2.10.

Inverse z-transform via power series

This is easily accomplished by long division when $X(z)$ is expressed as a ratio of polynomial in z. This method will not result in a closed form solution, but it is useful for plotting. In this method we perform the long hand division.

Example 2.12

Find the inverse z-transform of the discrete function in Example 2.8

$$X(z) = \frac{3z}{(z-1)(z-0.4)} = \frac{3z}{z^2 - 1.4z + 0.4}$$

Performing the division, we get

$$\begin{array}{r l}
 & 3z^{-1} + 4.2z^{-2} + 4.68z^{-3} + 4.872z^{-4} \\
z^2 - 1.4z + 0.4 \,\big) & 3z \\
 & \underline{3z - 4.2 + 1.2z^{-1}} \\
 & \quad 4.2 - 1.2z^{-1} \\
 & \quad \underline{4.2 - 5.88z^{-1} + 1.68z^{-2}} \\
 & \qquad 4.68z^{-1} - 1.68z^{-2} \\
 & \qquad \underline{4.68z^{-1} - 6.552z^{-2} + 1.872z^{-3}} \\
 & \qquad\qquad 4.872z^{-2} - 1.872z^{-3}
\end{array}$$

$$X(z) = 3z^{-1} + 4.2z^{-2} + 4.68z^{-3} + 4.872z^{-4} + \cdots$$

This is the same as the result obtained in Example 2.8. Long division can be accomplished in MATLAB using the command **dimpulse**.

Example 2.13

Using MATLAB find the inverse z-transform of the function in Example 2.12.

We use the following commands

```
num = [3 0];
den =[1 -1.4  0.4];
k =5;
xk = dimpulse(num, den, k)
```

The result is

```
xk =
        0
        3.0000
        4.2000
         4.6800
        4.8720
```

The above result is in agreement with the result in Example 2.12.

Example 2.14

Given the function $X(z) = \dfrac{0.8z^2}{(z-1)(z^2 - 0.4z + 0.2)}$ find the value of $x(kT)$ as k approaches infinity.

Since the function $(z-1)X(z)$ does not have any pole on or outside the unit circle in the z-plane, the final value theorem can be applied

$$\lim_{k\to\infty} x(kT) = \lim_{z=1} \frac{0.8z^2}{(z^2 - 0.4z + 0.2)} = 1$$

Solution of linear difference equation

The z-transform can be used to obtain the solution of the linear difference equation.

Example 2.15

Find the solution of the first-order zero-input difference

$$y(k+1) + y(k) = 0 \qquad \text{given } y(0) = 1$$

From the real translation operator (2.12), the z-transform is

$$zY(z) - zy(0) + Y(z) = 0$$

$$Y(z) = \frac{z}{z+1} y(0) = \frac{z}{z+1}(1)$$

Expanding $Y(z)$ into the power series

$$Y(z) = (1 - z^{-1} + z^{-2} - z^{-3} + z^{-4} + \cdots)y(0)$$
$$= (-1)^k z^{-k} y(0) \qquad k = 0,\ 1,\ 2,\ \cdots$$

$$Y^*(s) = (1 - e^{-T} + e^{-2T} - e^{-3T} + e^{-4T} + \cdots)y(0)$$
$$= (-1)^k e^{-kT} y(0) \qquad k = 0,\ 1,\ 2,\ \cdots$$

$$y^*(t) = \delta(t) - \delta(t-T) + \delta(t-2T) - \delta(t-3T) + \cdots$$

Example 2.16

Find the solution of the second order-order difference equation

$$y(k+2) - 4y(k+1) + 3y(k) = \delta(k)$$

With $y(0) = 1,\ y(1) = 0$ and $\delta(k)$ the discrete impulse function. Taking the z-transform, using the real translation relations (2.12, 2.13), we obtain

$$z^2Y(z) - z^2y(0) - zy(1) - 4[zY(z) - zy(0)] + 3Y(z) = 1$$

Substituting for $y(0)$ and $y(1)$ we have

$$[z^2 - 4z + 3]Y(z) = z^2 - 4z + 1$$

or

$$Y(z) = \frac{z^2 - 4z}{z^2 - 4z + 3} + \frac{1}{z^2 - 4z + 3} = \frac{z^2 - 4z + 1}{z^2 - 4z + 3}$$

Finding $\dfrac{Y(z)}{z}$

$$\frac{Y(z)}{z} = \frac{z^2 - 4z + 1}{z(z^2 - 4z + 3)} = \frac{z^2 - 4z + 1}{z(z-1)(z-3)} = \frac{\frac{1}{3}}{z} + \frac{1}{z-1} + \frac{-\frac{1}{3}}{z-3}$$

Therefore

$$Y(z) = \frac{1}{3} + \frac{z}{z-1} - \frac{1}{3}\frac{z}{z-3}$$

$$f(kT) = \frac{1}{3}\delta(t) - 1 - \frac{1}{3}(3)^{kT}$$

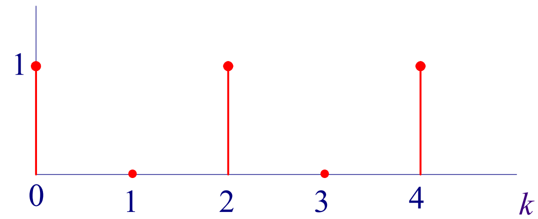
Example 2.17 (Example 11.6 Textbook)

Find the solution of following difference equation in power series form, then use z-transform to find a closed form solution.

$$m(k) = e(k) - e(k-1) - m(k-1) \qquad K \geq 0$$

where

$$e(k) = \begin{cases} 1 & k \text{ even, i.e.,} \quad k = 0, 2, 4, \cdots \\ 0 & k \text{ odd, i.e.,} \quad k = 1, 3, 5, \cdots \end{cases}$$



and

$e(-1) = 0$, and $m(-1) = 0$

(a) Solution is obtained by finding the successive values of $m(k)$ for $k = 0,\ 1,\ 2,\ \cdots$ i.e.,

$$m(0) = e(0) - e(-1) - m(-1) = 1 - 0 - 0 = 1$$
$$m(1) = e(1) - e(0) - m(0) = 0 - 1 - 1 = -2$$
$$m(2) = e(2) - e(1) - m(1) = 1 - 0 + 2 = 3$$
$$m(3) = e(3) - e(2) - m(2) = 0 - 1 - 3 = -4$$
$$m(4) = e(4) - e(3) - m(3) = 1 - 0 + 4 = 5$$

This method can be used efficiently if it is computed using digital computer.
In MATLAB you can use the following simple command.

```
ek = 1; ek_1 = 0; mk_1 = 0;
for n=1:5
        mk = ek - ek_1 - mk_1;
        k(n)=n-1; e(n)=ek; m(n)=mk;
        mk_1 = mk;
        ek_1 = ek;
        ek = 1 -ek;
end
disp(['    k     e     m'])
disp([k'   e' m'])
```

The result is

```
k    e    m
 0    1    1
 1    0   -2
2    1    3
 3    0   -4
 4    1    5
```

(b) The z-transform solution. Using the real translation operator (2.10), the z-transform of the above equation is

$$M(z) = E(z) - z^{-1}E(z) - z^{-1}M(z)$$

or

$$M(z) = \frac{1 - z^{-1}}{1 + z^{-1}} E(z) = \frac{z - 1}{z + 1} E(z)$$

Now $E(z)$ is given by

$$E(z) = 1 + z^{-2} + z^{-4} + \cdots = \left.\frac{1}{1-x}\right|_{x=z^{-2}} = \frac{1}{1-z^{-2}} = \frac{z^2}{z^2-1}$$

Therefore

$$M(z) = \frac{z-1}{z+1}\frac{z^2}{z^2-1}$$

We can obtain the power series form by long division, which will result in the same solution as before, i.e.,

$$M(z) = 1 - 2z^{-1} + 3z^{-2} - 4z^{-3} + 5z^{-4} + \cdots$$

MATLAB has many functions for finding the response of a discrete time system when the transfer function or state variable model is known. These are **dinitial, dstep, dimpulse**, and **dlsim**. Without any left hand argument the response plot is obtaine. If left hand argument is used with any of the above function, the values at discrete intervals are returned.

Example 2.18

Consider the z-domain transfer function given by

$$Y(z) = G(z)U(z)$$

where

$$G(z) = \frac{0.1z^{-1} + 0.25z^{-2}}{1 - 0.8z^{-1} + 0.15z^{-2}} = \frac{0.1z + 0.25}{1 - 0.8z + 0.15}$$

and $U(z)$ is the z-transform of a step input, i.e., $U(z) = \frac{z}{(z-1)}$

(a) Determine the final value of the step response
(b) Find the inverse z-transform y(k)
(c) Use MATLAB **dstep** function to find the step response.

$$Y(z) = \frac{0.1z + 0.25}{1 - 0.8 + 0.15}\frac{z}{z-1} = \frac{(0.1z + 0.25)z}{(z-1)(z-0.5)(z-.3)}$$

Roots of $G(z)$ are 0.5 and 0.3, which lie inside the unit circle, therefore the response has a final value

$$y(\infty) = \lim_{z\to 1}(z-1)Y(z) = \lim_{z\to 1}\frac{(0.1z+0.25)z}{(z-0.5)(z-0.3)} = \frac{0.1+0.25}{(0.5)(0.7)} = 1$$

(b) Performing the partial fraction expansion

$$\frac{Y(z)}{z} = \frac{0.1z + 0.25}{(z-1)(z-0.5)(z-0.3)} = \frac{1}{z-1} + \frac{-3}{z-0.5} + \frac{2}{z-0.3}$$

or

$$Y(z) = \frac{z}{z-1} + \frac{-3z}{z-0.5} + \frac{2z}{z-0.3}$$

The inverse z-transform is

$$y(k) = 1 - 3(0.5)^k + 2(0.3)^k$$

i.e.,

$$y(0) = 1 - 3(.5)^0 + 2(0.3)^0 = 0$$

$$y(1) = 1 - 3(.5)^1 + 2(0.3)^1 = 0.1$$

$$y(2) = 1 - 3(.5)^2 + 2(0.3)^2 = 0.43$$

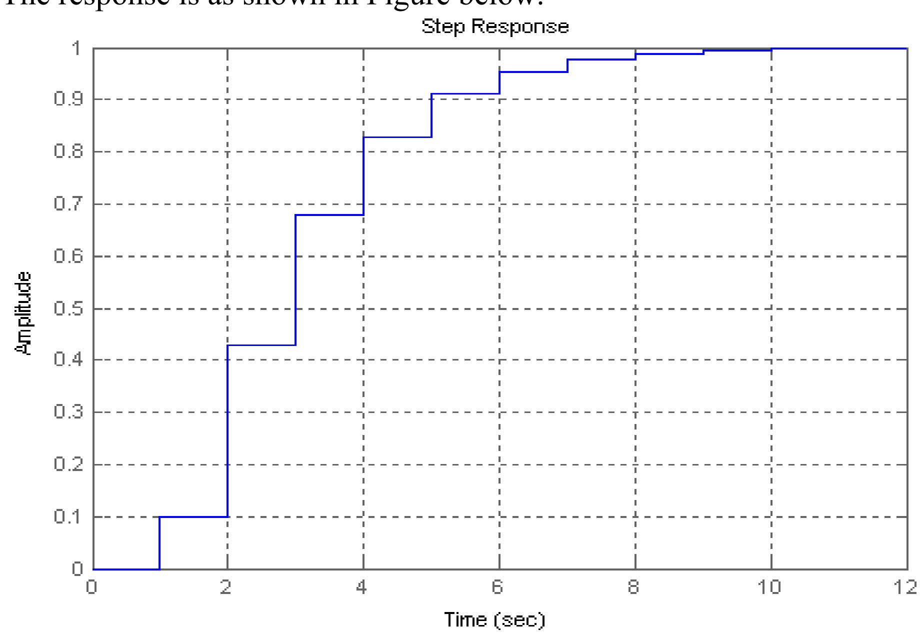
$$y(3) = 1 - 3(.5)^3 + 2(0.3)^3 = 0.679$$

$$y(0) = 1 - 3(.5)^4 + 2(0.3)^4 = 0.8287$$

(c) We use the following commands

```
num = [0.1  0.25];
den = [1  -0.8  0.15];
dstep(num, den)       % Step response plot
yk=dstep(num, den)    % Returns discrete values
```

The response is as shown in Figure below.



and the discrete values are

yk =

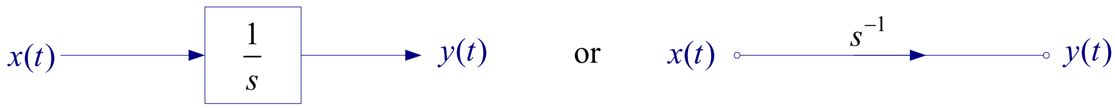
0
0.1000
0.4300
0.6790
0.8287
0.9111
0.9546
0.9770
0.9884
0.994

Simulation diagram and signal flow diagram

In the case of continuous system we have dealt with simulation diagram, which is a pictorial representation of the differential equation in time domain. The basic element of a continuous system simulation diagram is the integrator. That is the relation

$$y(t)=\int_0^t x(t)$$

is represented pictorially as



Note that the simulation diagram is in time domain and the operator $\frac{1}{s}$, or s^{-1} is simply the notation for the integrator.

Consider the following continuous time-domain equation

$$m(t)=e(t)-\int_0^t e(t)-\int_0^t m(t)$$

The simulation diagram of the following equation in continuous system is

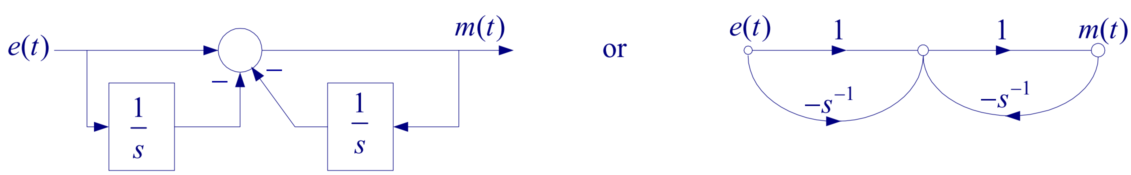


Figure 2.2 Continuous system simulation diagram.

We can apply Mason's gain formula to find the s-domain transfer function

$$G(s)=\frac{M(s)}{E(s)}=\frac{1-s^{-1}}{1+s^{-1}}=\frac{s-1}{s+1}$$

The basic element in discrete simulation diagram is a memory device known as shift register as shown in Figure 2.3(a). Suppose that every T seconds a number is shifted into the register, and at that instant, the number that was stored in the register is shifted out. Let $e(k)$ represent the number shifted into the register at an instant, then at that instant, the number shifted out is $e(k-1)$. The symbolic representation of this time delay or memory and shifting device is shown in the simulation diagram in Figure 2.3(b), along with the z-domain simulation diagram in Figure 2.3(c).

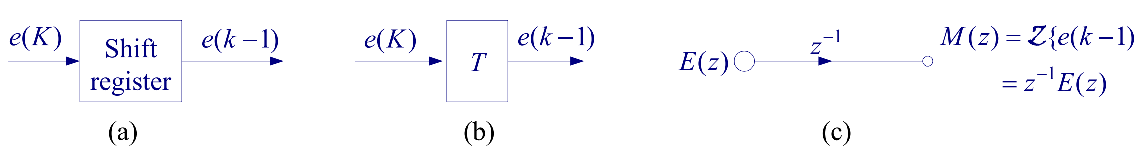


Figure 2.3 Ideal time-delay element.

Consider the difference equation given in Example 2.17 (Example 11.6 Textbook)

$$m(k) = e(k) - e(k-1) - m(k-1)$$

In an analogous manner to the continuous system, we draw the discrete simulation diagram using the time-delay or memory and shifting.

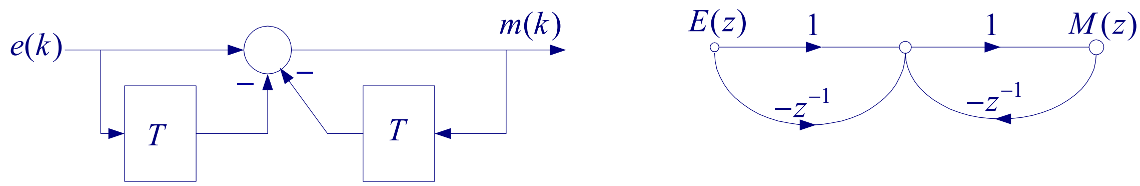


Figure 2.4 Simulation diagram and signal flow graph for the given example

We can apply Mason's gain formula to the discrete simulation diagram

$$G(z) = \frac{M(z)}{E(z)} = \frac{1 - z^{-1}}{1 + z^{-1}} = \frac{z-1}{z+1}$$

State variable model

Given a discrete transfer function, we can use the simulation diagram to obtain the state variable model of the system. Consider the following transfer function

$$G(z) = \frac{Y(z)}{U(z)} = \frac{b_1 z^{-1} + b_0 z^{-2}}{1 + a_1 z^{-1} + a_0 z^{-2}} = \frac{b_1 z + b_0}{z^2 + a_1 z + a_0}$$

Just like in the case of continuous-time system, there are different state space model depending on how the state variables are selected. We obtain the control canonical programming form simulation diagram. To do this we rewrite the above transfer function in cascade form

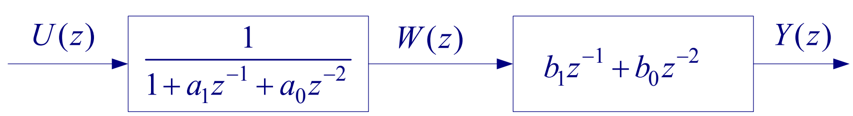


Figure 2.5 Cascade block diagram of the transfer function

Assigning the variable $W(z)$ at the output of the first block, we have

$$W(z) = U(z) - a_0 z^{-2} W(z) - a_1 z^{-1} W(z) \quad \text{and} \quad Y(z) = b_0 z^{-2} W(z) + b_1 z^{-1} W(z)$$

Utilizing two time delay we can draw the simulation diagram

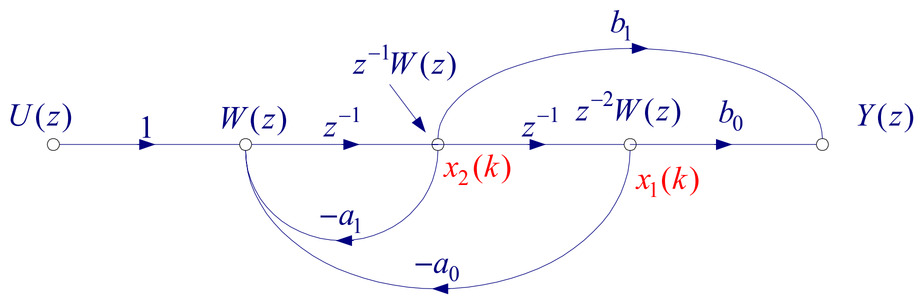


Figure 2.6 Control canonical programming form simulation diagram

The next step is to assign a state variable to the output of each time delay, and write an equation for the input of each time delay, which results in the following state equation

$$x_1(k+1) = x_2(k)$$
$$x_2(k+1) = -a_0 x_1(k) - a_1 x_2(k) + u(k)$$

and the output equation is

$$y(k) = b_0 x_1(k) + b_1 x_2(k)$$

or in matrix form

$$\begin{aligned} \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(k) &= [b_0 \quad b_1] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \end{aligned} \tag{2.18}$$

The generalized standard form of the state equation is written as

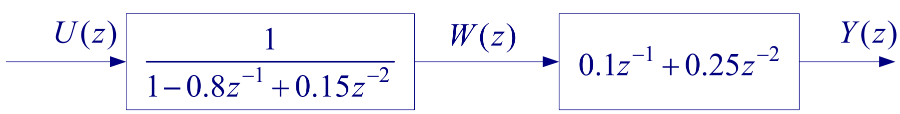
$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{A}\mathbf{x}(k) + \mathbf{B}u(k) \\ y(k) &= \mathbf{C}\mathbf{x}(k) + \mathbf{D}u(k) \end{aligned} \tag{2.19}$$

Example 2.19

Obtain the state variable model for the following z-domain transfer function

$$\frac{Y(z)}{U(z)} = \frac{0.1z^{-1} + 0.25z^{-2}}{1 - 0.8z^{-1} + 0.15z^{-2}} = \frac{0.1z + 0.25}{z^2 - 0.8z + 0.15}$$

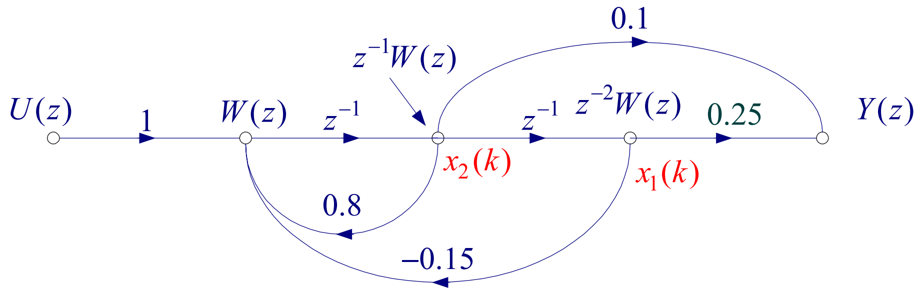
The cascade block diagram of the above transfer function is



Therefore

$W(z) = U(z) + 0.8z^{-1}W(z) + 0.15z^{-2}W(z)$ and $Y(z) = 0.1z^{-1}W(z) + 0.25z^{-2}W(z)$

which leads to the following simulation diagram



Writing the state equation from the simulation diagram, we have

$$x_1(k+1) = x_2(k)$$
$$x_2(k+1) = -0.15x_1(k) + 0.8x_2(k) + u(k)$$

and the output equation is

$$y(k) = 0.25x_1(k) + 0.1x_2(k)$$

Writing in matrix form we have,

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.15 & 0.8 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$
$$y(k) = \begin{bmatrix} 0.25 & 0.1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

MATLAB control system toolbox **[A, B, C, D]=dtf2ss(num, den)** obtains the state space model from the transfer function. For discrete-time transfer functions, you must make the length of the numerator and denominator equal to ensure correct results.

For the above example, we use the following commands

```
num = [0  0.1  0.25];    % For discrete TF, length of the numerator and
                         % denominator must be equal to ensure correct results.
den = [1  -0.8  0.15];
[A, B, C, D]=dtf2ss(num, den)
```

The result is

```
A =                                  B =
        0.8000   -0.1500                   1
        1.0000         0                   0
C =
        0.1000    0.25                D =
                                           0
```

Note that MATLAB assigns $x_1(k)$ to the output of the first shift register and $x_2(k)$ to the output of the second shift register.

Solution of the state equation

The state equation of discrete systems

$$\mathbf{x}(k+1) = \mathbf{x}(k) + \mathbf{B}u(k) \tag{2.20}$$

can be solved in power-series form or closed form using z-transformation technique.

a. Power-series solution

Given the input $u(k)$ and the initial states $\mathbf{x}(0)$, equation (2.20) is solved successively for $\mathbf{x}(k)$, $k = 1, 2, \cdots, n$ as follows:

$$\begin{aligned}
\mathbf{x}(1) &= \mathbf{A}\mathbf{x}(0) + \mathbf{B}u(0) \\
\mathbf{x}(2) &= \mathbf{A}\mathbf{x}(1) + \mathbf{B}u(1) = \mathbf{A}[\mathbf{A}\mathbf{x}(0) + \mathbf{B}u(0)] + \mathbf{B}u(1) \\
&= \mathbf{A}^2\mathbf{x}(0) + \mathbf{A}\mathbf{B}u(0) + \mathbf{B}u(1) \\
\mathbf{x}(3) &= \mathbf{A}\mathbf{x}(2) + \mathbf{B}u(2) = \mathbf{A}[\mathbf{A}^2\mathbf{x}(0) + \mathbf{A}\mathbf{B}u(0) + \mathbf{B}u(1)] + \mathbf{B}u(2) \\
&= \mathbf{A}^3\mathbf{x}(0) + \mathbf{A}^2\mathbf{B}u(0) + \mathbf{A}\mathbf{B}u(1) + \mathbf{B}u(2) \\
&\vdots \\
\mathbf{x}(n) &= \mathbf{A}^n\mathbf{x}(0) + \mathbf{A}^{n-1}\mathbf{B}u(0) + \mathbf{A}^{n-2}\mathbf{B}u(1) + \cdots + \mathbf{A}\mathbf{B}u(n-2) + \mathbf{B}u(n-1)
\end{aligned} \tag{2.21}$$

or the general solution is given by

$$\mathbf{x}(n) = \mathbf{A}^n\mathbf{x}(0) + \sum_{k=0}^{n-1} \mathbf{A}^{n-1-k}\mathbf{B}u(k) \tag{2.22}$$

b. z-transform solution of state equation

Taking the z-transform of equation (2.2), we get

$$z[\mathbf{X}(z) - \mathbf{x}(0)] = \mathbf{A}\mathbf{X}(z) + \mathbf{B}u(z)$$

or

$$[z\mathbf{I} - \mathbf{A}]\mathbf{X}(z) = z\mathbf{x}(0) + \mathbf{B}u(z)$$

Solving for the z-domain state variables, we get

$$\mathbf{X}(z) = \underbrace{z[z\mathbf{I} - \mathbf{A}]^{-1}\mathbf{x}(0)}_{\text{zero-input response}} + \underbrace{[z\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}u(z)}_{\text{zero-state response}} \tag{2.23}$$

Defining the state transition matrix as

$$\Phi(k) = \mathcal{Z}^{-1}\{z[z\mathbf{I} - \mathbf{A}]^{-1}\} = \mathbf{A}^k$$

The discrete-time solution can be expressed as

$$\mathbf{x}(n) = \Phi(n)\mathbf{x}(0) + \sum_{k=0}^{n-1} \Phi(n-1-k)\mathbf{B}u(k) \tag{2.24}$$

MATAB control system toolbox has several functions for the solution state equation. dinitial(A, B, C, D, X0) plots the time response of the discrete system due to an initial condition on the states. The number of sample points is automatically determined based on the system poles and zeros. dinitial(A, B, C, D, X0, N) uses the user-supplied number of points, N. When invoked with left hand arguments [Y,X, N] = dinitial(A,B,C,D,X0,...) returns the output and state responses (Y and X), and the number of points (N). No plot is drawn on the screen. The matrix Y has as many columns as outputs and X has as many columns as there are states. dstep(A,B,C,D,IU) plots the response of the discrete system to a step applied to the single input IU when there is one input IU is set to 1. The number of points is determined automatically. dstep(A,B,C,D,IU,N) uses the user-supplied number of points, N. When invoked with left hand arguments, [Y,X] = dstep(A,B,C,D,...) returns the output and state time history in the matrices Y and X. No plot is drawn on the screen. Y has as many columns as there are outputs and X has as many columns as there are states. Also functions dimpulse and dlsim are used for impulse response and response to user defined discrete function.

Example 2.20

The state variable model of the system in Example 2.19 is

$$\mathbf{x}(k+1) = \begin{bmatrix} 0 & 1 \\ -0.15 & 0.8 \end{bmatrix}\mathbf{x}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(k) \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$y(k) = \begin{bmatrix} 0.25 & 0.1 \end{bmatrix}\mathbf{x}(k)$$

Where $u(k)$ is a discrete unit step input.

(a) Find the zero-input response
(b) Find the zero-state response

$$\begin{bmatrix} z\mathbf{I} - \mathbf{A} \end{bmatrix} = \begin{bmatrix} z & -1 \\ 0.15 & z-0.8 \end{bmatrix}$$

$$\begin{bmatrix} z\mathbf{I} - \mathbf{A} \end{bmatrix}^{-1} = \frac{\begin{bmatrix} z-0.8 & 1 \\ -0.15 & z \end{bmatrix}}{z^2 - 0.8z + 0.15} = \frac{\begin{bmatrix} z-0.8 & 1 \\ -0.15 & z \end{bmatrix}}{(z-0.5)(z-0.3)}$$

The zero-input solution is

$$\mathbf{X}(z) = z[z\mathbf{I}-\mathbf{A}]^{-1}\mathbf{x}(0)$$

Therefore,

$$\mathbf{x}(z) = \frac{z\begin{bmatrix} z-0.8 & 1 \\ -0.15 & z \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}}{(z-0.5)(z-0.3)} = \frac{z\begin{bmatrix} z-0.8 \\ -0.15 \end{bmatrix}}{(z-0.5)(z-0.3)}$$

Therefore,

$$\frac{\mathbf{x}(z)}{z} = \begin{bmatrix} \dfrac{z-0.8}{(z-0.5)(z-0.3)} \\ \dfrac{-0.15}{(z-0.5)(z-0.3)} \end{bmatrix} = \begin{bmatrix} \dfrac{-0.15}{(z-0.5)} + \dfrac{2.5}{(z-0.3)} \\ \dfrac{-0.75}{(z-0.5)} + \dfrac{0.75}{(z-0.3)} \end{bmatrix}$$

or

$$\mathbf{x}(z) = \begin{bmatrix} \dfrac{-0.15z}{(z-0.5)} + \dfrac{2.5z}{(z-0.3)} \\ \dfrac{-0.75z}{(z-0.5)} + \dfrac{0.75z}{(z-0.3)} \end{bmatrix}$$

Taking the inverse z-transform, the solution is

$$x_1(k) = -1.5(0.5)^k + 2.5(0.3)^k$$
$$x_2(k) = -0.75(0.5)^k + 0.75(0.3)^k$$

and the output response is

$$y(k) = 0.25x_1(k) + 0.1x_2(k)$$
$$y(k) = -0.45(0.5)^k + 0.7(0.3)^k$$

For $k = 0,1,2,\cdots$ the discrete-time values are

$$y(0) = -0.45(0.5)^0 + 0.7(0.3)^0 = 0.25$$
$$y(1) = -0.45(0.5)^1 + 0.7(0.3)^1 = -0.015$$
$$y(2) = -0.45(0.5)^2 + 0.7(0.3)^2 = -0.0495$$
$$y(3) = -0.45(0.5)^3 + 0.7(0.3)^3 = -0.0374$$
$$y(4) = -0.45(0.5)^4 + 0.7(0.3)^4 = -0.0225$$

(b) The z-domain zero-state response is given by

$$\mathbf{X}(z) = [z\mathbf{I}-\mathbf{A}]^{-1}\mathbf{B}u(z)$$

Therefore,

$$\mathbf{X}(z) = \frac{\begin{bmatrix} z-0.8 & 1 \\ -0.15 & z \end{bmatrix}\begin{bmatrix} 0 \\ \overline{1} \end{bmatrix}}{(z-0.5)(z-0.3)}\frac{z}{(z-1)} = \frac{z\begin{bmatrix} 1 \\ z \end{bmatrix}}{(z-1)(z-0.5)(z-0.3)}$$

or

$$\frac{\mathbf{X}(z)}{z} = \begin{bmatrix} \dfrac{1}{(z-1)(z-0.5)(z-.3)} \\ \dfrac{z}{(z-1)(z-0.5)(z-.3)} \end{bmatrix} = \begin{bmatrix} \dfrac{1/0.35}{z-1} + \dfrac{-10}{z-0.5} + \dfrac{1/0.14}{z-0.3} \\ \dfrac{1/0.35}{z-1} + \dfrac{-5}{z-0.5} + \dfrac{0.3/0.14}{z-0.3} \end{bmatrix}$$

Thus

$$\mathbf{X}(z) = \begin{bmatrix} \dfrac{1/0.35z}{z-1} + \dfrac{-10z}{z-0.5} + \dfrac{1/0.14z}{z-0.3} \\ \dfrac{1/0.35z}{z-1} + \dfrac{-5z}{z-0.5} + \dfrac{0.3/0.14z}{z-0.3} \end{bmatrix}$$

Taking the inverse z-transform results in

$$x_1(k) = \frac{1}{0.35} - 10(0.5)^k + \frac{1}{0.14}(0.3)^k$$

$$x_2(k) = \frac{1}{0.35} - 5(0.5)^k + \frac{0.3}{0.14}(0.3)^k$$

and

$$y(k) = 0.25x_1(k) + 0.1x_2(k)$$

therefore

$$y(k) = 1 - 3(0.5)^k + 2(0.3)^k$$

For $k = 0, 1, 2, \cdots$ the discrete-time values are

$$y(0) = 1 - 3(0.5)^0 + 2(0.3)^0 = 0$$

$$y(1) = 1 - 3(0.5)^1 + 2(0.3)^1 = 0.1$$

$$y(2) = 1 - 3(0.5)^2 + 2(0.3)^2 = 0.43$$

$$y(3) = 1 - 3(0.5)^3 + 2(0.3)^3 = 0.679$$

$$y(4) = 1 - 3(0.5)^4 + 2(0.3)^4 = 0.8287$$

The complete solution is the sum of zero-input response and zero-state response.

Example 2.21 (chd2ex21.m)

Use the MATLAB control system functions **dinitial** and **dstep** to obtain the zero-input and zero-state response for the system in Example 2.20.

We use the following commands

```
A = [0  1; -0.15  0.8];
B = [0;  1];
C = [0.25  0.1];
D = 0;
Disp('Zero-input response or natural response')
x0 = [1; 0];
figure(1), dinitial(A, B, C, D, x0), grid  % Step response plot
[yk, x] = dinitial(A, B, C, D, x0,5)       % Returns zero-input responses y, x1, x2
```

% for 5 points

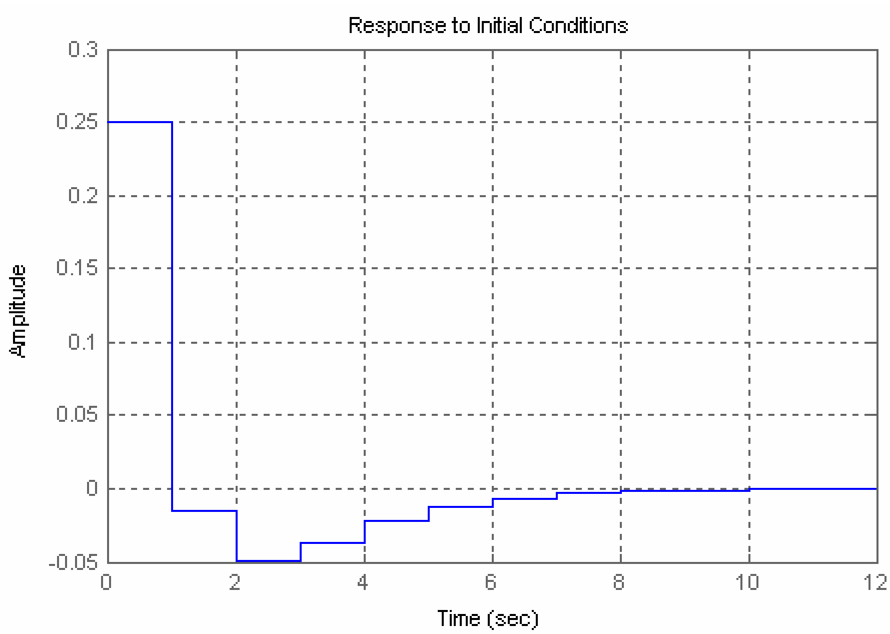
```
disp('Zero-state response or steady-state response')
figure(2), dstep(A, B, C, D,1), grid        % Step response plot
[yk, x] = dstep(A, B, C, D, 1,  5)    % Returns step responses y, x1, x2 for 5 points
```

The result is

Zero-input response or natural response

```
yk =
        0.2500
        -0.0150
         -0.0495
        -0.0374
         -0.0225
x =

         1.0000          0
        0                 -0.1500
        -0.1500          -0.1200
         -0.1200         -0.0735
         -0.0735         -0.0408
```

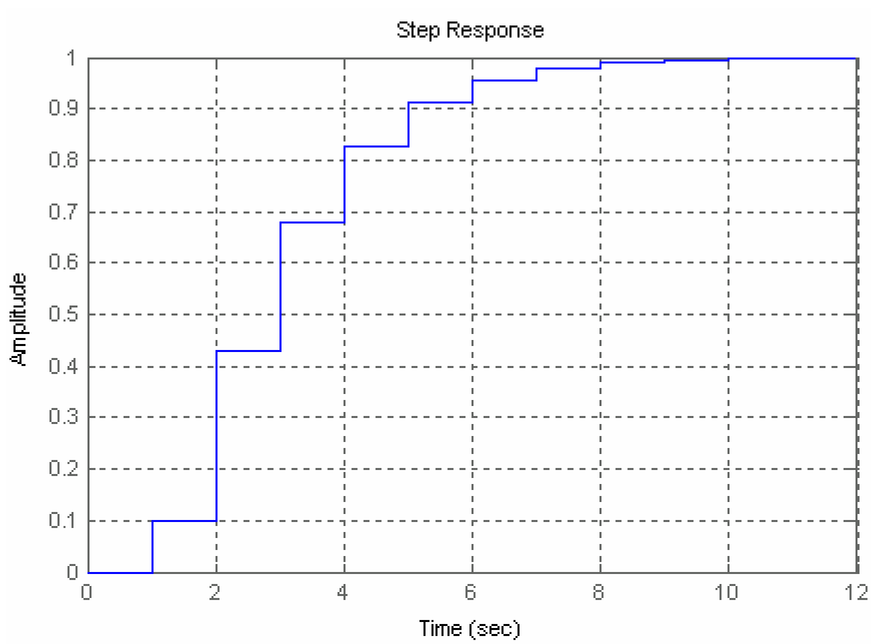


Zero-state response or steady-state response yk =

0
0.1000
0.4300
0.6790
0.8287

x =

0	0
0	1.0000
1.0000	1.8000
1.8000	2.2900
2.2900	2.5620



For both cases the MATLAB results are identical with the analytical computations.